Stability in the Problem of Selecting Project Portfolios with Multiple Criteria

Vladimir Korotkov

University of Turku

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$$\begin{aligned} \min \quad & f(x, C) = (f_1(x, C_1), f_2(x, C_2), \ldots, f_s(x, C_s)), \\ \text{s. t.} \quad & x \in X, \end{aligned}$$

where $s \geq 2$; $x = (x_1, x_2, \ldots, x_n)^T \in X \subset \mathbb{Z}^n$, $n \in \mathbb{N}$; $C_k \in \mathbb{R}^{m \times n}$ is the k-th cut of $C = [c_{ijk}] \in \mathbb{R}^{m \times n \times s}$.

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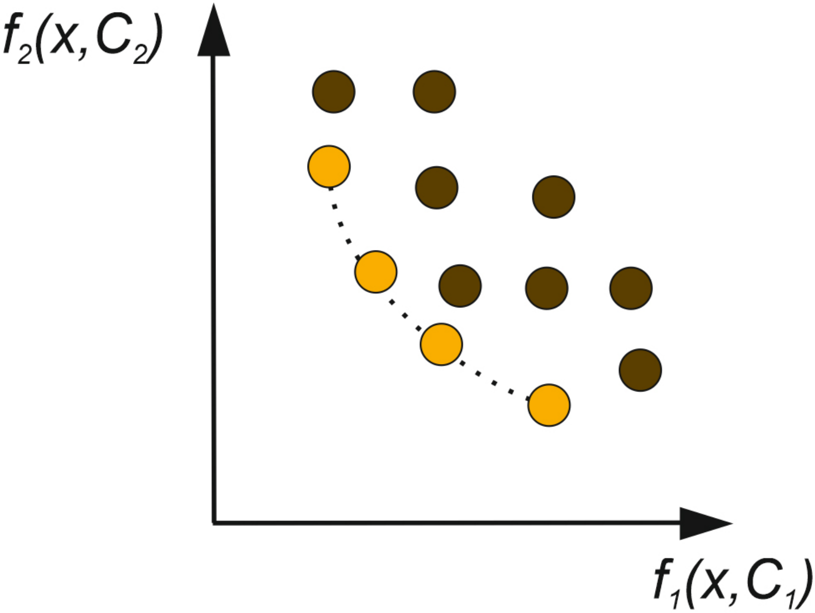
$$P^s(C) = \{x \in X : \ \nexists x' \in X \ \ (f(x, C) \geq f(x', C) \ \& \ f(x, C) \neq f(x', C))\}.$$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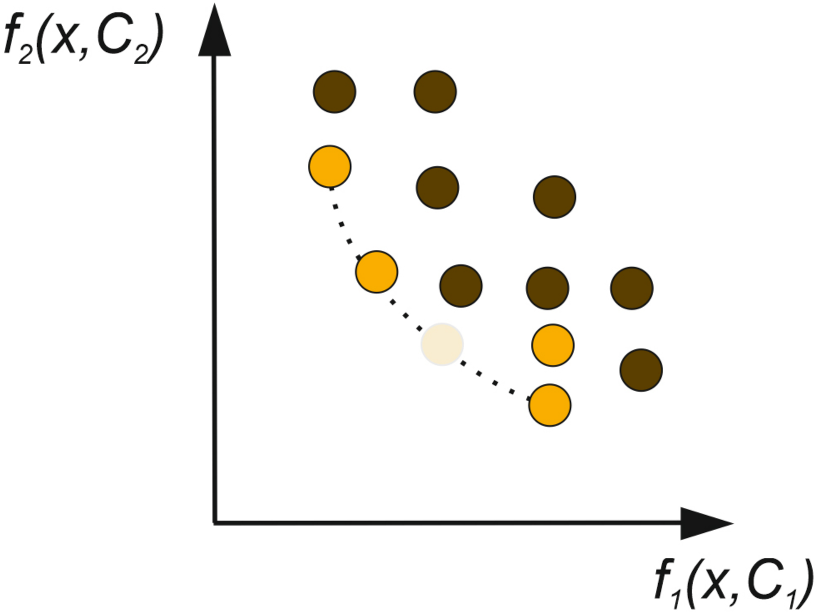
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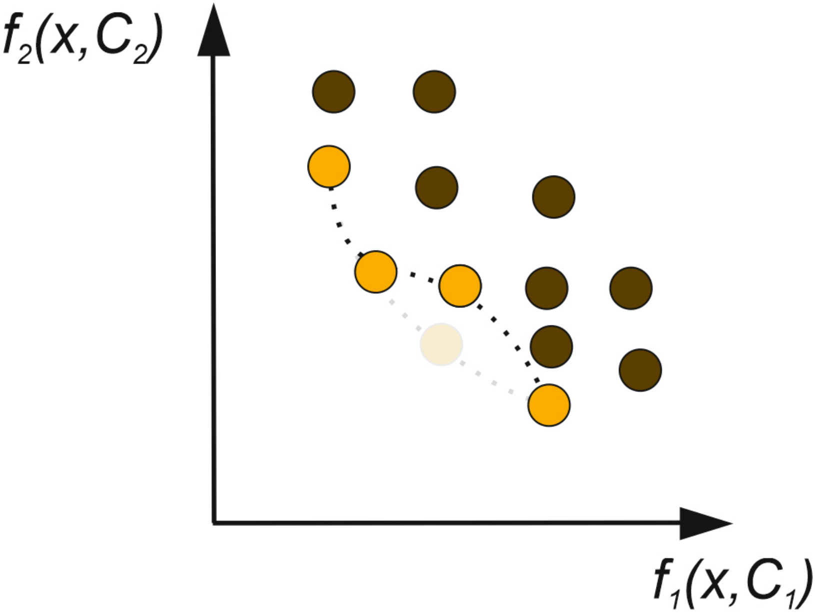
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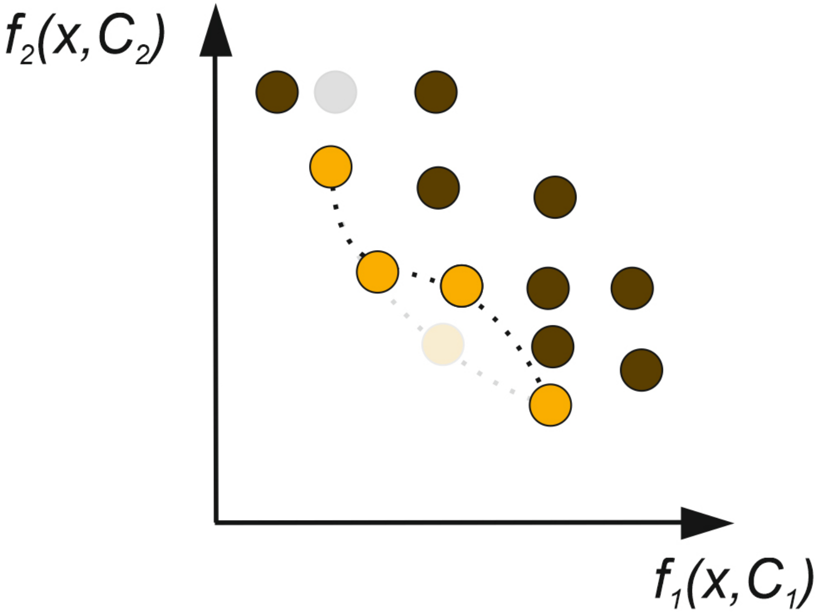

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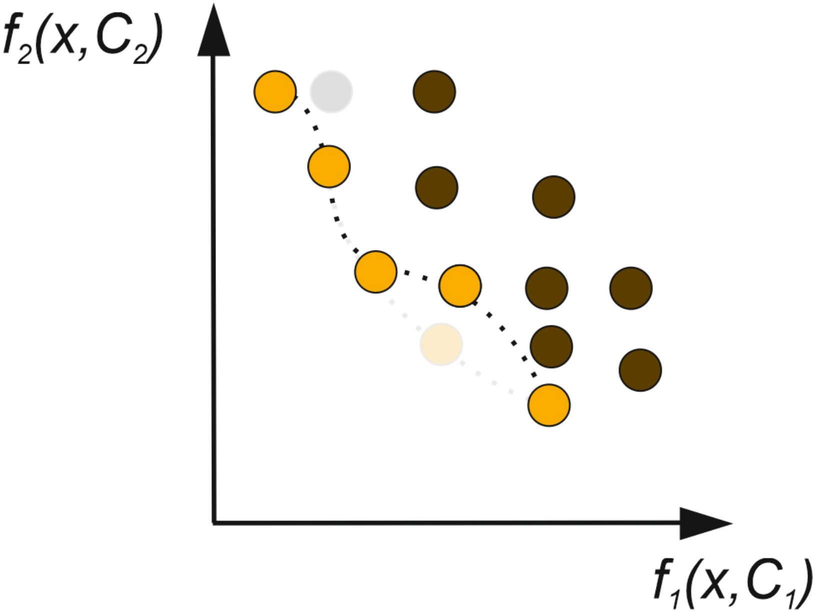

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$x_j = 0$ otherwise;

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e_{ijk} is efficiency of the k-th type, evaluating project $j \in N_n$ in the case, when the market is in the state $i \in N_m = \{1, 2, \ldots, m\}$;
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For each market state $i \in N_m$ and each risk type $k \in N_s$, investment portfolio $x \in X$ is evaluated by index of efficiency and risk (additive functions):

$$\sum_{j \in N_n} e_{ijk} x_j \quad \text{and} \quad \sum_{j \in N_n} r_{ijk} x_j.$$

Savage's criteria

$$f_k(x, R_k) = \max_{i \in N_m} R_{ik}x = \max_{i \in N_m} \sum_{j \in N_n} r_{ijk}x_j \to \min_{x \in X}, \qquad k \in N_s,$$

where $R_k \in \mathbb{R}^{m \times n}$ is the k-th cut $R = [r_{ijk}] \in \mathbb{R}^{m \times n \times s}$ with rows $R_{ik} = (r_{i1k}, r_{i2k}, \ldots, r_{ink}) \in \mathbb{R}^n$, $i \in N_m$.

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$$f_k(x, E_k) = \min_{i \in N_m} E_{ik} x = \min_{i \in N_m} \sum_{j \in N_n} r_{ijk} x_j \to \max_{x \in X}, \qquad k \in N_s,$$

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Criteria of extreme optimism

$$f_k(x, E_k) = \max_{i \in N_m} E_{ik} x = \max_{i \in N_m} \sum_{j \in N_n} e_{ijk} x_j \to \max_{x \in X}, \qquad k \in N_s.$$

$$\|C\|_{ppp} = \Big\| \Big(\|C_1\|_{pp}, \|C_2\|_{pp}, \ldots, \|C_s\|_{pp} \Big) \Big\|_p -$$

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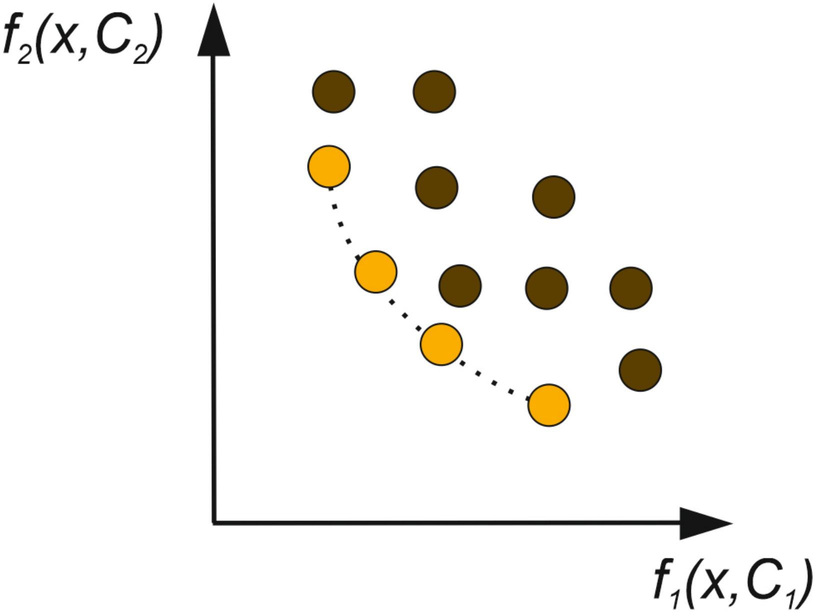
$$\|a\|_p = \begin{cases} \Big(\sum\limits_{k \in N_s} |a_k|^p \Big)^{1/p} & \text{if } \ 1 \le p < \infty, \\ \\ \max\{|a_k| : \ k \in N_s\} & \text{if } \ p = \infty, \end{cases}$$

$$a = (a_1, a_2, \ldots, a_s) \in \mathbb{R}^s.$$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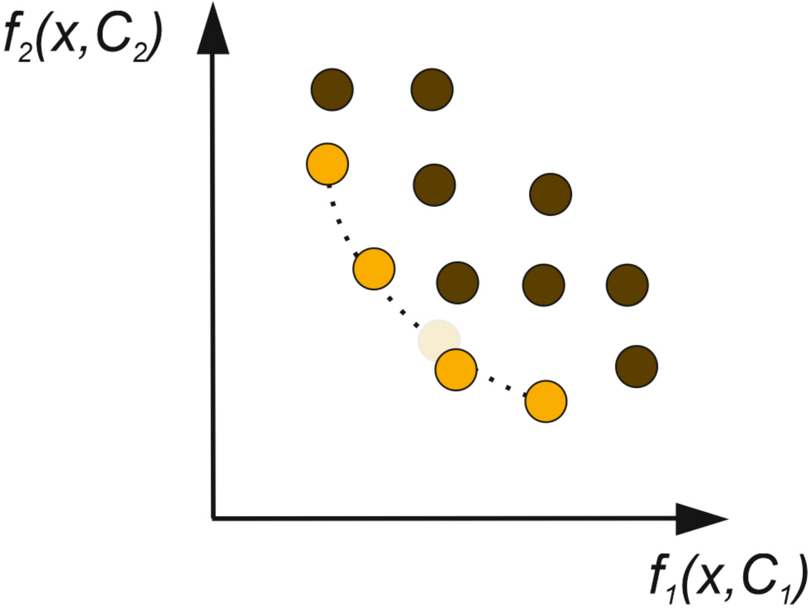
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The stability radius of a Pareto-optimal portfolio x^0:

$$\rho^s(x^0, C, p, p, p) = \begin{cases} \sup\ \Xi & \text{if } \ \Xi \neq \emptyset, \\ 0 & \text{if } \ \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \{\varepsilon > 0 : \ \forall C' \in \Omega(\varepsilon) \quad (x^0 \in P^s(C + C')\},$$

$$\Omega(\varepsilon) = \{C' \in \mathbb{R}^{m \times n \times s} : \quad \|C'\|_{ppp} < \varepsilon\}.$$

Theorem 1

Let

$$\varphi_1 = \min_{x \in X \setminus \{x^0\}} \max_{k \in N_s} \min_{i^0 \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i^0 k}x^0),$$

$$\varphi_2 = \min_{x \in X \setminus \{x^0\}} \sum_{k \in N_s} [\min_{i' \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i' k}x^0)]^+,$$

where

$$[a]^+ = \max\{0, a\},$$

then

$$\varphi_1 \leq \rho^s(x^0, R, 1, 1, \infty) \leq m\varphi_1,$$

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Theorem 2

Let

$$\varphi_3 = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x,x^0,R)]^+\|_\infty}{\|x\|_{p'} + \|x^0\|_{p'}},\ \psi_3 = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x,x^0,R)]^+\|_\infty}{\|x - x^0\|_{p'}},$$

$$\varphi_4 = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x,x^0,R)]^+\|_\infty}{\|x + x^0\|_1},\ \psi_4 = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x,x^0,R)]^+\|_\infty}{\|x - x^0\|_1},$$

$$\varphi_5 = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x,x^0,R)]^+\|_p}{\|x + x^0\|_1},\ \psi_5 = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x,x^0,R)]^+\|_p}{\|x - x^0\|_1},$$

where $g(x,x^0,R) = f(x,R) - f(x^0,R)$, $1/p + 1/p' = 1$, then

$$\varphi_3 \le \rho^s(x^0, R, p, \infty, \infty) \le \psi_3,$$

$$\varphi_4 \le \rho^s(x^0, R, \infty, p, \infty) \le m^{1/p}\psi_4,$$

$$\varphi_5 \le \rho^s(x^0, R, \infty, \infty, p) \le \psi_5.$$

The stability radius of a Pareto-optimal portfolio x^0:

$$\rho^2(x^0, E, R, \infty, \infty, \infty) = \begin{cases} \sup\ \Xi & \text{if } \ \Xi \neq \emptyset, \\ 0 & \text{if } \ \Xi = \emptyset, \end{cases}$$

$$\Xi = \{\varepsilon > 0 : \quad \forall (E', R') \in \Omega(\varepsilon) \quad (x^0 \in P(E + E', R + R'))\},$$

$$\Omega(\varepsilon) = \{(E', R') \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} : \quad \max\{\|E'\|_{\infty\infty}, \|R'\|_{\infty\infty}\} < \varepsilon\}.$$

Theorem 3

Let

$$\varphi_6 = \min_{x \in X \setminus \{x^0\}} \frac{\gamma(x^0, x)}{\|x^0 + x\|_1}, \ \psi_6 = \min_{x \in X \setminus \{x^0\}} \frac{\gamma(x^0, x)}{\|x^0 - x\|_1},$$

where

$$\gamma(x^0, x) = \max\Big\{ \max_{i \in N_m} \min_{i' \in N_m} (E_{i'} x^0 - E_i x), \ \min_{i \in N_m} \max_{i' \in N_m} (R_{i'} x - R_i x^0) \Big\},$$

then

$$\varphi_6 \leq \rho^2(x^0, E, R, \infty, \infty, \infty) \leq \psi_6.$$

The stability radius of the problem:

$$\rho^s(C,p,p,p) = \begin{cases} \sup\ \Xi & \text{if } \ \Xi \neq \emptyset, \\ 0 & \text{if } \ \Xi = \emptyset, \end{cases}$$

where

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Theorem 4

Let

$$\varphi_7 = \min_{x \notin P^s(E)} \max_{x' \in P^s(x,E)} \min_{k \in N_s} \max_{i \in N_m} \min_{i' \in N_m} (E_{i'k}x' - E_{ik}x),$$

$$\varphi_8 = \min_{x \notin P^s(R)} \max_{x' \in P^s(x,R)} \min_{k \in N_s} \min_{i' \in N_m} \max_{i \in N_m} \frac{R_{ik}x - R_{i'k}x'}{\|x + x'\|_1},$$

$$\psi_8 = \min_{x \notin P^s(R)} \max_{x' \in P^s(x,R)} \min_{k \in N_s} \min_{i' \in N_m} \max_{i \in N_m} \frac{R_{ik}x - R_{i'k}x'}{\|x - x'\|_1},$$

then

$$\varphi_7 \le \rho^s(E, 1, 1, \infty) \le mn\varphi_7,$$

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then

$$\varphi_7 \le \rho^s(E, 1, 1, \infty) \le mn\varphi_7,$$

$$\varphi_8 \le \rho^s(R, \infty, \infty, \infty) \le \psi_8.$$

The stability radius of the problem:

$$\rho^2(E, R, p, p, \infty) = \begin{cases} \sup \ \Xi & \text{if} \ \ \Xi \neq \emptyset, \\ 0 & \text{if} \ \ \Xi = \emptyset, \end{cases}$$

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Theorem 5

Let

$$\varphi_9 = \min_{x \notin P(E,R)} \max_{x' \in P(x,E,R)} \frac{\gamma(x,x')}{\|(\|x\|_q, \|x'\|_q)\|_q},$$

$$\psi_9 = \min_{x \notin P(E,R)} \max_{x' \in P(x,E,R)} \frac{\gamma(x,x')}{\|x - x'\|_1},$$

$$\gamma(x,x') = \min\{f_1(x',E) - f_1(x,E),\ f_2(x,R) - f_2(x',R)\},$$

then

$$\varphi_9 \le \rho^2(E,R,p,p,\infty) \le (mn)^{1/p}\psi_9.$$

Example

Let $m = 2,\ n = 3,$
$X = \{x^1, x^2, x^3\}, \quad x^1 = (1, 1, 0),\ x^2 = (1, 0, 1),\ x^3 = (0, 1, 1),$

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the matrices $E \in \mathbb{R}^{2\times 3}$ and $R \in \mathbb{R}^{2\times 3}$:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

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Then

$$f(x^1) = (1,1),\ f(x^2) = (2,1),\ f(x^3) = (3,2),$$

$$P(E,R) = \{x^2, x^3\},$$

$$x^1 \notin P(E,R).$$

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$$\varphi = \psi = 0.$$

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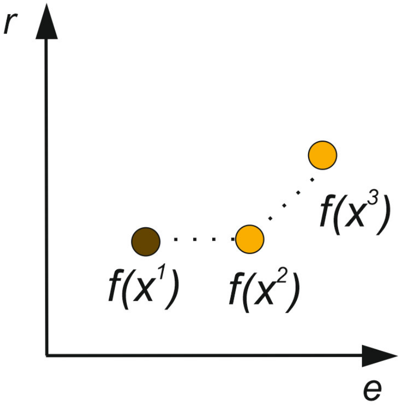

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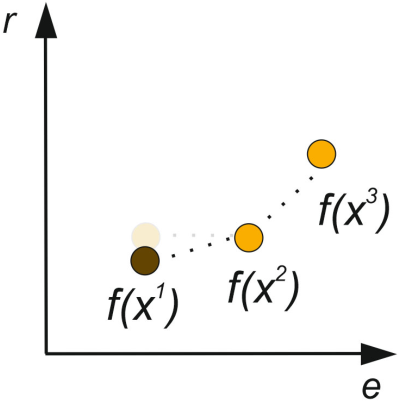

$$f(x^1) = (1, 1 - \varepsilon),$$

$$f(x^2) = (2, 1),$$

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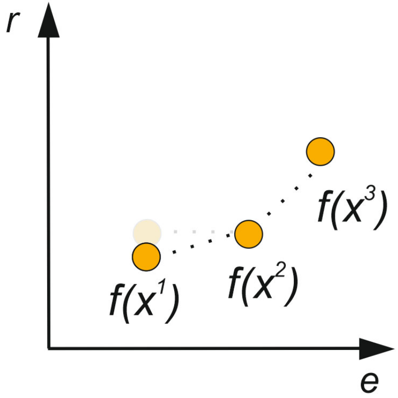
$$P(E, R) = \{\mathbf{x}^1, x^2, x^3\}.$$

Example



$$f(x^1) = (1, 1-\varepsilon),$$

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the matrices $E \in \mathbb{R}^{2\times 3}$ and $R \in \mathbb{R}^{2\times 3}$:

$$E = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}.$$

Then

$$f(x^1) = (1,4), \; f(x^2) = (2,3), \; f(x^3) = (3,4),$$
$$P(E,R) = \{x^2, x^3\},$$
$$x^1 \notin P(E,R).$$

Example

Let $m = 2, \ n = 3$,
$X = \{x^1, x^2, x^3\}, \quad x^1 = (1,1,0), \ x^2 = (1,0,1), \ x^3 = (0,1,1)$,
the matrices $E \in \mathbb{R}^{2\times 3}$ and $R \in \mathbb{R}^{2\times 3}$:

$$E = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}.$$

Then

$$f(x^1) = (1,4), \ f(x^2) = (2,3), \ f(x^3) = (3,4),$$
$$P(E,R) = \{x^2, x^3\},$$
$$x^1 \notin P(E,R).$$

For $1 \leq p \leq \infty$

$$\varphi = \max\{\frac{1}{4^{1-1/p}}, 0\} = \frac{1}{4^{1-1/p}},$$
$$\psi = \max\{\frac{1}{2}, 0\} = \frac{1}{2}.$$

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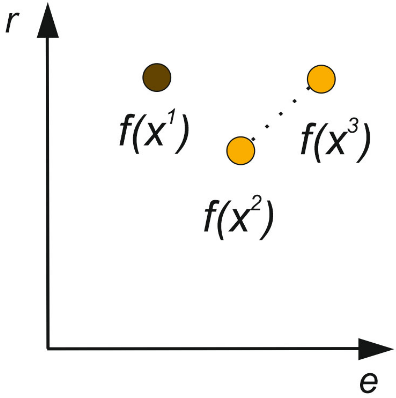
For $p = \infty$

$$\varphi = 1/4,$$
$$\psi = 1/2.$$

Example



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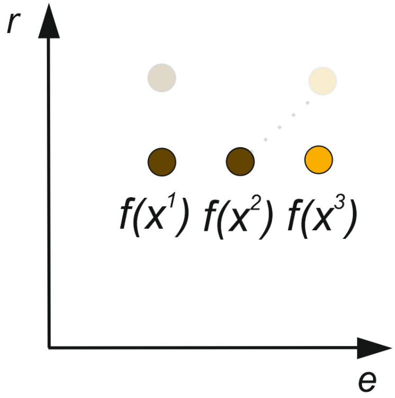
Then

$$f(x^1) = (1, \mathbf{3}),\ f(x^2) = (2, 3),\ f(x^3) = (3, \mathbf{3}),$$

$$P(E + E^0, R + R^0) = \{x^3\},$$

$$x^1, x^2 \notin P(E + E^0, R + R^0).$$

Example



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Then

$$f(x^1) = (1, \mathbf{3} - \varepsilon), \; f(x^2) = (2,3), \; f(x^3) = (3, \mathbf{3} + \varepsilon),$$

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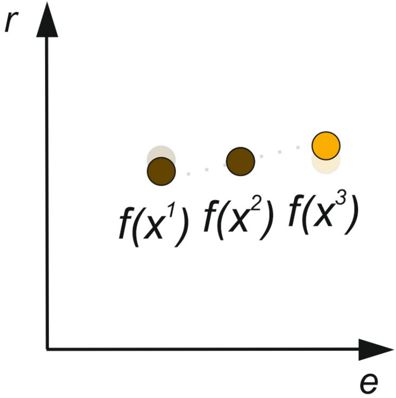
Example



$$f(x^1) = (1, \mathbf{3} - \varepsilon),$$

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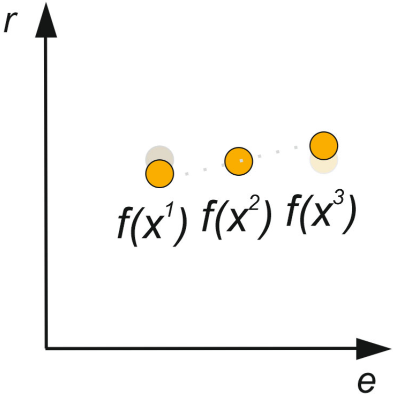

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Thank you for your attention!