$$\overbrace{\textbf{ESH}}^{\text{EXTENDED SUPPORTING HYPERPLANE}}$$

THE ESH ALGORITHM FOR CONVEX MINLP

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Andreas Lundell
Jan Kronqvist
Tapio Westerlund

Optimization and Systems Engineering
Åbo Akademi University, Finland

CONTENTS OF THE TALK

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The extended supporting hyperplane (ESH) algorithm:

- Solves sequences of approximative LP and MILP problems.
- Cutting planes are replaced with supporting hyperplanes generated on the boundary of the feasible set.
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- Two LP preprocessing steps are utilized to quickly get a tight linear relaxation of the feasible set.

Comparison of a COIN-OR ESH implementation to other solvers.

THE ECP ALGORITHM

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Implemented, *e.g.*, in the AlphaECP solver in GAMS and available on the NEOS server.

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Kelley Jr. J. The cutting-plane method for solving convex programs, Journal of the SIAM, vol. 8(4), 703–712.

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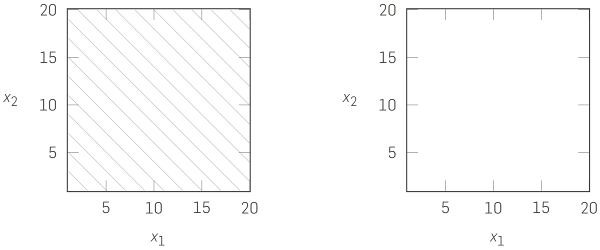
AN ILLUSTRATIVE EXAMPLE



$$
\begin{aligned}
\text{minimize} \quad & c^T x = -x_1 - x_2 \\
\text{subject to} \quad & g_1(x_1, x_2) = 0.15(x_1 - 8)^2 + 0.1(x_2 - 6)^2 + 0.025 e^{x_1} x_2^{-2} - 5 \le 0 \\
& g_2(x_1, x_2) = 1/x_1 + 1/x2 - x_1^{0.5} x_2^{0.5} + 4 \le 0 \\
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& 1 \le x_1 \le 20, \quad 1 \le x_2 \le 20, \quad x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{Z}.
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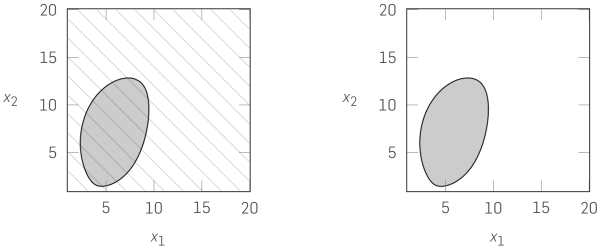

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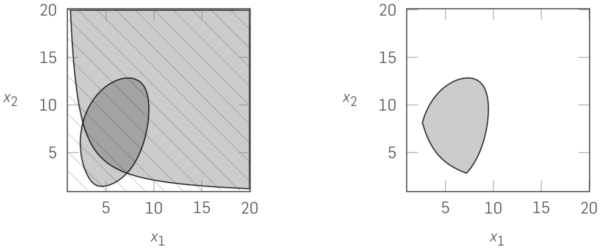

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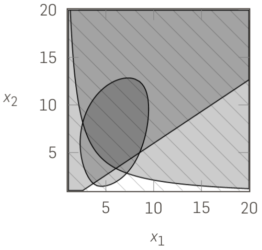

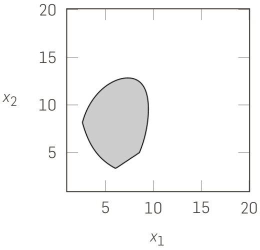

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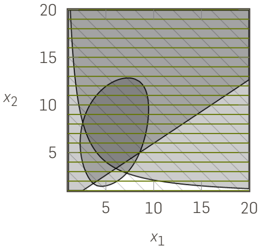

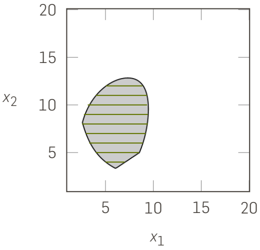

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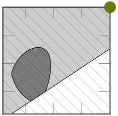

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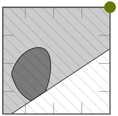
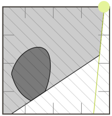


Select the constraint with the largest error $i = \arg\max g_i(x_1^k, x_2^k)$ and generate a cutting plane

$$g_i(x_1^k, x_2^k) + \nabla g_i(x_1^k, x_2^k)^T (x - x_1^k, x - x_2^k) \leq 0.$$



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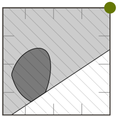

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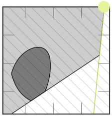
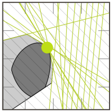


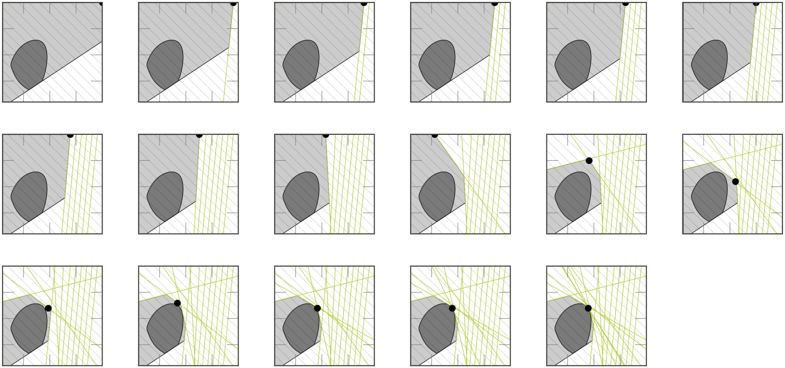
If the solution of the MILP problem is feasible for the MINLP problem *i.e.*,

$$g_i(x_1^k, x_2^k) \leq \epsilon \quad \forall i = 1, 2,$$

the optimal solution has been found.



To solve the problem with the ECP algorithm ($\epsilon = 0.001$) it takes 17 iterations (17 MILP problems solved to optimality).



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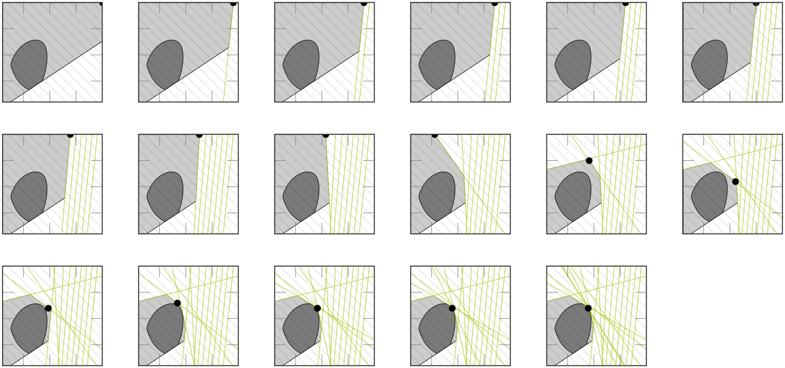

How can we improve performance?

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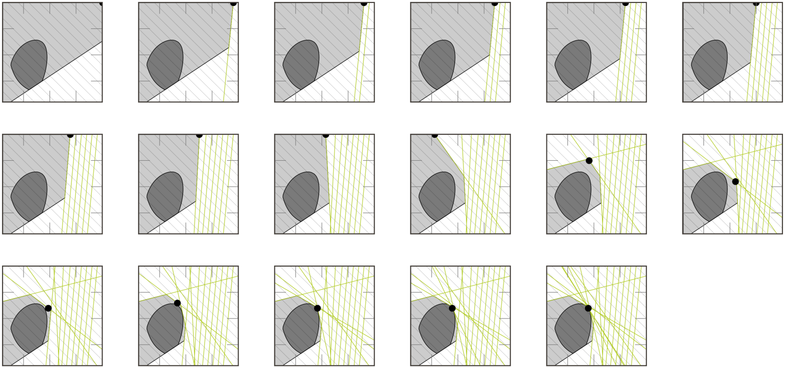

How can we improve performance?

Generate cutting planes on the boundary of the feasible set!

THE ESH ALGORITHM

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A new, stable and efficient algorithm for solving convex MINLP problems to global optimality.

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Similar ideas as presented in:

The supporting hyperplane method for unimodal programming, Veinott Jr. A. F., Operations Research, Vol. 15, pp. 147–152, 1967

The ESH algorithm is briefly described in:

An extended supporting hyperplane algorithm for convex MINLP problems, Lundell A., Kronqvist J. and Westerlund T. Proceedings of XII Global Optimization Workshop, pp. 21–24, 2014

THE MINLP PROBLEM SCOPE

The ESH algorithm solves convex MINLP problems of the type

$$x^* = \underset{x \in C \cap L \cap Y}{\arg\min}\, c^T x \tag{P}$$

where $x = [x_1, x_2, \dots, x_N]^T$ belongs to the compact set

$$X = \{x \mid \underline{x}_i \leq x_i \leq \overline{x}_i,\ i = 1, \dots, N\} \subset \mathbb{R}^n,$$

the feasible region is defined by $C \cap L \cap Y$, where

$$\begin{aligned} C &= \{x \mid g_m(x) \leq 0,\ m = 1, \dots, M,\ x \in X\} \\ L &= \{x \mid Ax \leq a,\ Bx = b,\ x \in X\} \\ Y &= \{x \mid x_i \in \mathbb{Z},\ i \in I_{\mathbb{Z}},\ x \in X\} \end{aligned}$$

and C is a convex set.

BREAKDOWN OF THE ESH ALGORITHM

NLP step

Obtain a feasible, relaxed interior point (satisfying C) by solving a NLP problem.

LP1 step (optional)

Solve simple LP problems (initially in X) to obtain supporting hyperplanes.

LP2 step (optional)

As in LP1 but also include the linear constraints in L.

MILP step

Solve MILP problems to find the optimal solution to (P).

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- A point in C is required as an endpoint for the line searches.

NLP STEP

- A point in C is required as an endpoint for the line searches.
- Assuming (P) has a solution, the internal point can be obtained from the NLP problem:

$$\begin{aligned} \tilde{x}_{\text{NLP}} &= \operatorname*{arg\,min}_{x \in X} F(x), \\ \text{where } \; F(x) &:= \max_{m=1,\dots,M} \{g_m(x)\}. \end{aligned} \tag{P-NLP}$$

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- F is convex since it is the maximum of convex functions.
- (P-NLP) may be nonsmooth (if $M > 1$) even if g_m is smooth.
- The point $\tilde{x}_{\mathrm{NLP}}$ need not be optimal but then fulfill $F(\tilde{x}_{\mathrm{NLP}}) < 0$.

LP1 STEP

- Starting from $k = 1,\ \Omega_0 = X$, the problem

$$\tilde{x}^k_{\mathrm{LP}} = \arg\min_{\Omega_{k-1}}\ c^T x \tag{P-LP1}$$

is repeatedly solved, and supporting hyperplanes (SHs)

$$l_k := F(x^k) + \nabla_F(x^k)^T(x - x^k) \leq 0$$

are generated and added to Ω_k.

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  is repeatedly solved, and supporting hyperplanes (SHs)

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  are generated and added to Ω_k.
- A point x^k is obtained by a line search for $F(x^k) = 0$ between $\tilde{x}_{\mathrm{NLP}}$ and the solution to (P-LP1) $\tilde{x}^k_{\mathrm{LP}}$:

$$x^k = \lambda\tilde{x}_{\mathrm{NLP}} + (1 - \lambda)\tilde{x}^k_{\mathrm{LP}}, \quad \lambda \in [0, 1].$$

- If not $F(\tilde{x}^k_{\mathrm{LP}}) < \epsilon_{\mathrm{LP1}}$ or a maximum number of SHs have been generated, then k is increased and (P-LP1) resolved.

LP2 STEP

— This step is otherwise identical to LP1, with the exception that the linear constraints in L are now also included, *i.e.*,

$$\boxed{\tilde{x}_{\text{LP}}^{k} = \underset{\Omega_{k-1} \cap L}{\arg\min} \; c^T x} \qquad \text{(P-LP2)}$$

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— (P-LP2) is repeatedly solved until $F(\tilde{x}^k_{\mathrm{LP}}) < \epsilon_{\mathrm{LP2}}$ or a maximum number of SHs have additionally been generated.

— In many cases the LP1 step may be omitted.

MILP STEP

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- This step is similar to LP2, but the integer requirements in Y are also included, *i.e.*,

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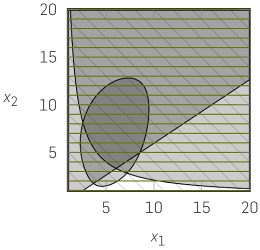
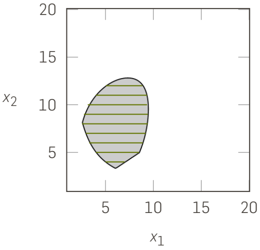
- (P-MILP) is repeatedly solved until $F(\tilde{x}^k_{\text{MILP}}) < \epsilon_{\text{MILP}}$.
- Intermediate (P-MILP) problems do not need to be solved to optimality, but in order to guarantee an optimal solution of (P), the final MILP solution must be optimal.





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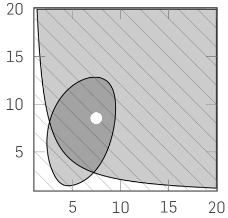
NLP STEP – FIND AN INTERIOR POINT

$$\tilde{x}_{\mathrm{NLP}} = \operatorname*{arg\,min}_{(x_1,x_2)\in X} F(x_1, x_2),$$

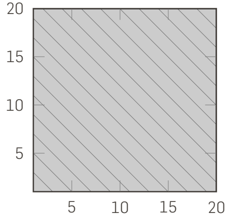
$$\text{where } F(x_1, x_2) := \max\{g_1(x_1, x_2),\ g_2(x_1, x_2)\}.$$

- The solution can be found using a suitable NLP solver
- Not required to be the optimal point
- The optimal point here is $(7.45, 8.54)$



LP1-STEP – ITERATION 1

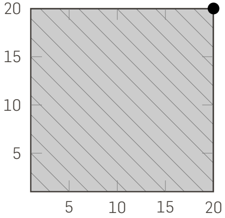
– Assume initially that $\Omega_0 = X$.



LP1-STEP – ITERATION 1

- Assume initially that $\Omega_0 = X$.
- $k = 1$, solve LP in Ω,

$$\tilde{x}^k_{\text{LP}} = \underset{\Omega_{k-1}}{\arg\min}\; c^T x.$$



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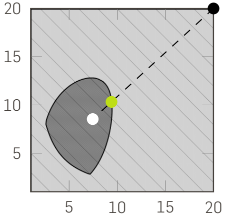

- Do line search

$$x^k = \lambda \tilde{x}_{\text{NLP}} + (1-\lambda)\tilde{x}^k_{\text{LP}}.$$

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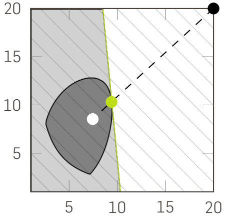


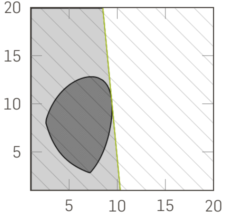
- Do line search

$$x^k = \lambda \tilde{x}_{\text{NLP}} + (1 - \lambda)\tilde{x}_{\text{LP}}^k.$$

- Generate supporting hyperplane in x^k and add to Ω.

– $\Omega_1 = \{x | l_1(x) \leq 0,\ x \in X\}$.

$$l_1(x) = 3.26x_1 + 0.313x_2 - 33.9$$



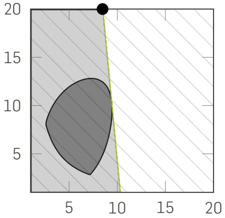
LP1-STEP – ITERATION 2

- $\Omega_1 = \{x | l_1(x) \leq 0,\ x \in X\}$.

$$l_1(x) = 3.26x_1 + 0.313x_2 - 33.9$$

- $k = 2$, solve LP in Ω,

$$\tilde{x}^k_{\text{LP}} = \arg\min_{\Omega_{k-1}}\ c^T x.$$



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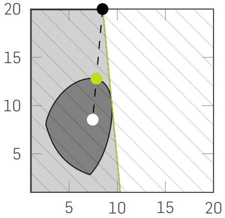

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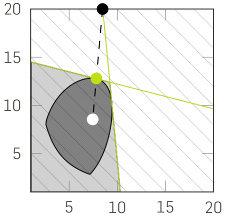
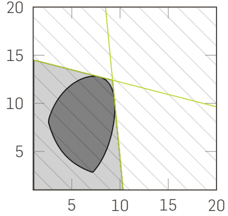


- Do line search $x^k = \lambda \tilde{x}_{\text{NLP}} + (1 - \lambda)\tilde{x}^k_{\text{LP}}$.
- Generate supporting hyperplane in x^k and add to Ω.

- $\Omega_2 = \{x | l_j(x) \leq 0,\ j \in \{1, 2\},\ x \in X\}$

$$l_1(x) = 3.26x_1 + 0.313x_2 - 33.9$$
$$l_2(x) = 0.332x_1 + 1.30x_2 - 19.2$$



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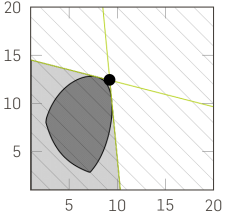

- $k = 3$, solve LP in Ω,

$$\tilde{x}_{\mathrm{LP}}^k = \arg\min_{\Omega_{k-1}}\ c^T x.$$

LP1-STEP – ITERATION 3

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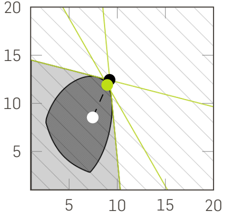

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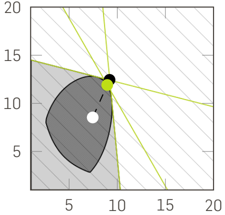

– $k = 3$, solve LP in Ω,

$$\tilde{x}^k_{\mathrm{LP}} = \arg\min_{\Omega_{k-1}} \ c^T x.$$

– Do line search, generate supporting hyperplane and add to Ω.

– Terminate LP1-step since $F(\tilde{x}^k_{\mathrm{LP}}) < \epsilon_{\mathrm{LP1}}$.

LP2-STEP – ITERATION 4

– $\Omega_3 = \{x | l_j(x) \leq 0,\ j \in \{1, 2, 3\},\ x \in X\}$

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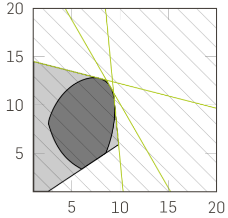
$$l_3(x) = 1.66x_1 + 0.951x_2 - 26.2$$



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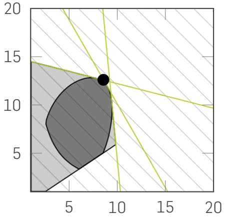
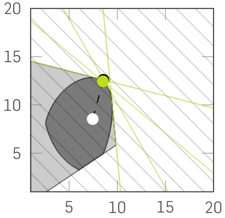


- $k = 4$, solve LP now in $\Omega \cap L$,

$$\tilde{x}^k_{\text{LP}} = \arg\min_{\Omega_{k-1} \cap L}\ c^T x.$$

- $\Omega_3 = \{x | l_j(x) \leq 0,\ j \in \{1, 2, 3\},\ x \in X\}$

$$\begin{aligned} l_1(x) &= 3.26x_1 + 0.313x_2 - 33.9 \\ l_2(x) &= 0.332x_1 + 1.30x_2 - 19.2 \\ l_3(x) &= 1.66x_1 + 0.951x_2 - 26.2 \end{aligned}$$



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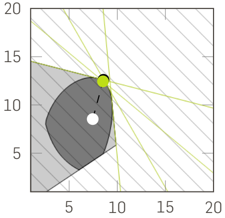

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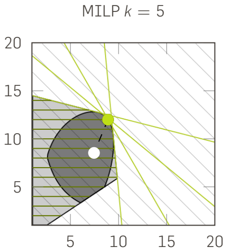
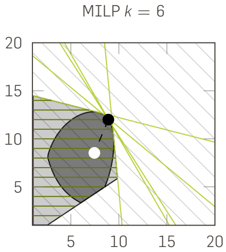
– Terminate LP2-step since $F(\tilde{x}^k_{\mathrm{LP}}) < \epsilon_{\mathrm{LP2}}$.

MILP STEP – ITERATIONS 5 AND 6





- In this step the integer requirements in Y are also considered, *i.e.*, initially $k = 5$, $\Omega = \Omega_{k-1} \cap L \cap Y$.
- The MILP steps are required to guarantee an integer-feasible solution.

— Solving the MINLP problem with the supporting hyperplane algorithm gives the following solution

Type	Iteration	Obj. funct.	x_1	x_2	$F(x_1, x_2)$
LP1	1	-40.0000	20.0000	20.0000	30 359
LP1	2	-28.4720	8.47199	20.0000	14.9321
LP1	3	-21.6378	9.19722	12.4406	0.957382
LP2	4	-21.1639	8.56022	12.6037	0.229455
MILP	5	-20.9065	8.90647	12	0.00442134
MILP	6	-20.9036	8.90362	12	$4.22619 \cdot 10^{-6}$

SOLUTION AND COMPARISONS TO OTHER SOLVERS

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- Solution times compared to some other MINLP solvers:

Solver	Subproblems solved	Time (s)	Implementation
ESH	4 LP + 2 MILP (2 OPT)	0.04	COIN-OR (IPOPT + CPLEX)
ECP (convex)	22 MILP (9 OPT) + 12 NLP	1.51	GAMS 24.2 (CONOPT + CPLEX)
DICOPT	11 NLP + 10 MILP	1.00	GAMS 24.2 (CONOPT + CPLEX)
SBB	4 NLP	0.41	GAMS 24.2 (CONOPT)

NUMERICAL COMPARISONS

SOME PRELIMINARY TESTS

A prototype ESH solver utilizing COIN-OR projects, IPOpt as NLP solver and CPLEX 12.6 as MILP solver was applied to some convex MINLP test problems from MINLPLib2.

It was compared to some MINLP solvers available in GAMS:

- AlphaECP (convex version), ANTIGONE, BARON, DICOPT, SBB and SCIP

The runs were terminated after 1,800 seconds.

QUADRATIC PROBLEMS

Solver	alan	fac3	netmod-dol2	netmod-kar1	slay05h
ESH	**0.01**	0.76	94.19	21.87	38.49
ECP	0.28	0.22	467.35	82.54	84.46
ANTIGONE	0.33	92.75	113.9	157.01	0.61
BARON	0.14	1.31	∗	#	1,067.14
DICOPT	0.14	0.53	#	#	**0.19**
SBB	**0.01**	**0.20**	#	23.26	6.13
SCIP	**0.01**	0.23	**43.57**	**4.04**	1.24
Variables	8	66	1,998	456	230
Binaries	4	12	462	136	40
Integers	0	0	0	0	0
Type	MBQP	MBQP	MBQP	MBQP	MBQP

Solver	du-opt	ex4
ESH	33.3	1.01
ECP	22.98	0.75
ANTIGONE	∗	**0.22**
BARON	13.74	2.62
DICOPT	#	0.44
SBB	**0.33**	1.06
SCIP	0.7	0.45
Variables	20	36
Binaries	0	25
Integers	13	0
Type	MIQP	MBQCQP

All times in seconds. Terminated problems after 1,800 s indicated with ∗. Nonoptimal solution indicated with #.

GENERAL NONLINEAR PROBLEMS

Solver	batchs101006m	enpro56pb	o7	rsyn0805m04h	rsyn0830m04h	sssd25-08
ESH	17.89	1.34	**461.75**	6.64	111.63	**58.71**
ECP	17.8	2.37	$*$	4.46	25.02	$*$
ANTIGONE	15.68	0.75	$*$	$*$	$*$	668.71
BARON	161.81	7.75	$*$	66.87	$*$	$*$
DICOPT	**1.75**	**0.34**	#	**2.31**	**4.57**	$*$
SBB	#	#	#	10.8	171.91	#
SCIP	11.68	1.48	$*$	13.63	$*$	$*$
Variables	278	127	114	1,400	1,956	256
Binaries	129	73	42	296	416	224
Integers	0	0	0	0	0	0
Type	MBNLP	MBNLP	MBNLP	MBNLP	MBNLP	MBNLP

Solver	fo7-ar4-1	fo9-ar3-1	jit1	m7-ar5-1
ESH	**32.51**	**169.15**	0.12	**1.06**
ECP	79.45	612.68	0.25	4.84
ANTIGONE	33.58	$*$	1.36	1.61
BARON	$*$	$*$	0.1	166.62
DICOPT	#	#	#	#
SBB	#	#	0.03	#
SCIP	34.12	393.22	**0.01**	13.15
Variables	112	180	25	112
Binaries	0	0	0	0
Integers	42	72	4	42
Type	MINLP	MINLP	MINLP	MINLP

All times in seconds. Terminated problems after 1,800 s indicated with $*$. Nonoptimal solution indicated with #.

CONCLUSIONS

FUTURE WORK

A journal paper will soon be submitted.

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- Mathematica / Wolfram Language
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Extensive comparisons to other solvers.

We are looking for additional reference test problems!

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Any questions?