Solving linearly constrained nonlinear minimax problems using cutting plane techniques

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Contents of the talk

► Brief introduction to the Extended Supporting Hyperplane (ESH) algorithm.
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  ▶ Subproblem in the ESH algorithm.
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- This subproblem can be solved as a continuous nonlinear minimax problem.
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- How to solve linearly constrained nonlinear minimax problems by cutting plane techniques.
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- Brief introduction to the Extended Supporting Hyperplane (ESH) algorithm.
  - Subproblem in the ESH algorithm.
- This subproblem can be solved as a continuous nonlinear minimax problem.
- How to solve linearly constrained nonlinear minimax problems by cutting plane techniques.
- Numerical comparison of different methods for a test set of minimax problems.
ESH is an algorithm intended for solving convex Mixed-Integer Nonlinear Programming (MINLP) optimization problems.¹

¹ The extended supporting hyperplane algorithm for convex mixed-integer nonlinear programming, Kronqvist, J., Lundell, A., Westerlund, T., Journal of Global Optimization (2015), Accepted
ESH is an algorithm intended for solving convex Mixed-Integer Nonlinear Programming (MINLP) optimization problems.\(^1\)

A subproblem in the ESH algorithm can be solved as a linearly constrained nonlinear minimax problem.

- Finding a suitable interior point of a convex set.

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\(^1\) The extended supporting hyperplane algorithm for convex mixed-integer nonlinear programming, Kronqvist, J., Lundell, A., Westerlund, T., Journal of Global Optimization (2015), Accepted
The MINLP problem

\[
\text{find } x^* \in \arg\min_{x \in L \cap C \cap Y} c^T x, \tag{P-MINLP}
\]

\[
X = \left\{ x \left| \underline{x}_i \leq x_i \leq \bar{x}_i, \ i = 1, \ldots, N , x \in \mathbb{R}^n \right. \right\},
\]

\[
L = \left\{ x \left| Ax \leq a, \ Bx = b, \ x \in X \right. \right\},
\]

\[
C = \left\{ x \left| g_m(x) \leq 0, \ m = 1, \ldots, M, x \in X \right. \right\},
\]

\[
Y = \left\{ x \left| x_i \in \mathbb{Z}, \ \forall i \in I_{\mathbb{Z}}, x \in X \right. \right\}.
\]
The MINLP problem

\[
\text{find } x^* \in \arg\min_{x \in L \cap C \cap Y} c^T x, \quad (P\text{-MINLP})
\]

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\]

The ESH algorithm solves \((P\text{-MINLP})\) by solving a sequence of linearly relaxed subproblems.
In each iteration the objective function is minimized within a linearly overestimated set $\Omega$ of the feasible region defined by $L \cap C \cap Y$. In case the current solution point $x_k$ is not within the set $C$, a supporting hyperplane is generated and added to set $\Omega$. The supporting hyperplane improves the linear relaxation of (P-MINLP) and excludes the current solution point $x_k$ from $\Omega$. In order to find the generation point for the supporting hyperplane, a point within the interior of $C$ is needed. This point is used for line searches for the boundary of the set $C$. 
In each iteration the objective function is minimized within a linearly overestimated set $\Omega$ of the feasible region defined by $L \cap C \cap Y$

In case the current solution point $x_k$ is not within the set $C$ (the nonlinear constraints are not satisfied).

- A supporting hyperplane to the set $C$ is generated and added to set $\Omega$.
- The supporting hyperplane improves the linear relaxation of (P-MINLP) and excludes the current solution point $x_k$ from $\Omega$. 
In each iteration the objective function is minimized within a linearly overestimated set $\Omega$ of the feasible region defined by $L \cap C \cap Y$.

In case the current solution point $x_k$ is not within the set $C$ (the nonlinear constraints are not satisfied).

- A supporting hyperplane to the set $C$ is generated and added to set $\Omega$.
- The supporting hyperplane improves the linear relaxation of (P-MINLP) and excludes the current solution point $x_k$ from $\Omega$.

In order to find the generation point for the supporting hyperplane a point within the interior of $C$ is needed.

- This point is used for line searches for the boundary of the set $C$. 
A sketch of the main principle of the ESH algorithm, a line search is conducted between the interior point and the current solution $x^k$ and a supporting hyperplane is generated.

$g(x) = 0$

$g(x) < 0$
A sketch of the main principle of the ESH algorithm, a line search is conducted between the interior point and the current solution $x^k$ and a supporting hyperplane is generated.

\[ g(x) = 0 \]

\[ g(x) < 0 \]

How does the choice of interior point affect the ESH algorithm?

How to obtain an interior point efficiently?
To illustrate how the choice of interior point affects the ESH algorithm, consider the following MINLP problem:

minimize \[-0.05x_1 - 15x_2\]
subject to \[g_1(x_1, x_2) = 0.1x_1^2 + 0.05x_2^2 - 0.025x_1x_2 - 90 \leq 0\]
\[g_2(x_1, x_2) = 0.5x_1^2 + 0.35x_2^2 + 2x_1 - 45x_2 + 130 \leq 0\]
\[-30 \leq x_1 \leq 45, -20 \leq x_2 \leq 55,\]
\[x_1 \in \mathbb{R}, \quad x_2 \in \mathbb{Z}.\]  

(EX1)
Contours of the objective function, the feasible region defined by the nonlinear constraints and the region defined by the linear constraints (variable bounds).
The figures shows the first iteration of the ESH algorithm with two different interior points.
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The figures show the second iteration of the ESH algorithm with two different interior points.
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Depending on the interior point it either takes 9 or 6 iterations with ESH algorithm to solve the MINLP problem.
Desired properties of the interior point:
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- Force the relaxed solution into the feasible set with as few supporting hyperplanes as possible.
- Easy to obtain!
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What about the Chebyschev center?
A Chebychev center of the convex set $C$ is a point inside the set farthest from the exterior of $C$, i.e., the deepest point of the set $C$. \(^2\)

Let's consider a case where the set $C$ is defined as

$$C = \{ x \mid g_1(x) \leq 0, \ g_2(x) \leq 0 \}$$

$$g_1(x) = 0.1x_1^2 + 0.05x_2^2 - 0.025x_1x_2 - 90$$

$$g_2(x) = 0.5x_1^2 + 0.35x_2^2 + 2x_1 - 45x_2 + 130$$

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\(^2\) Convex Optimization, Boyd, S., and Vandenberghe, L., Cambridge University Press
A Chebychev center of the convex set $C$ is a point inside the set farthest from the exterior of $C$, i.e., the deepest point of the set $C$. \(^3\)

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$$g_2(x) = 0.5 x_1^2 + 0.35 x_2^2 + 2 x_1 - 45 x_2 + 130$$
The Chebyshev center of the set
\[ C = \{ x \mid g_m(x) \leq 0 \quad \forall m, x \in \mathbb{R}^n \} \]
can be obtained by solving the following convex optimization problem

\[
\text{maximize} \quad R \\
\text{subject to} \quad \tilde{g}_m(x, R) \leq 0 \quad \forall m
\]

where \( \tilde{g}_m(x, R) := \sup_{||u|| \leq 1} g_m(x + Ru) \).

\[ (1) \]

However, even if this is a convex NLP problem it is difficult to solve unless the constraint functions \( \tilde{g}_m \) can be determined analytically.

Finding the Chebyshev center on the previous slide required the solution of 266 convex NLP problems!
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(1)

\[ u \in \mathbb{R}^n. \]
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\[ \text{Convex Optimization, Boyd, S., and Vandenberghe, L., Cambridge University Press} \]
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However, the only requirement for an interior point $\bar{x}$ with the ESH algorithm is:

$$\max\{g_1(\bar{x}), g_2(\bar{x}), \ldots, g_M(\bar{x})\} < 0.$$
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However, the only requirement for an interior point $\bar{x}$ with the ESH algorithm is:

$$\max \{ g_1(\bar{x}), g_2(\bar{x}),..., g_M(\bar{x}) \} < 0.$$ 

An interior point for the ESH algorithm can therefore be found by solving the following problem:

\[
\text{find } \bar{x} \in \arg \min_{x \in L} F(x), \\
\text{where } F(x) := \max \{ g_1(x), g_2(x),..., g_M(x) \}.
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An interior point for the ESH algorithm can therefore be found by solving the following problem:

$$\text{find } \bar{x} \in \arg\min_{x \in L} F(x),$$

(2)

where $F(x) := \max\{g_1(x), g_2(x), \ldots, g_M(x)\}$. 
The linearly constrained minimax problem

\[
\text{find } \bar{x} \in \arg\min_{x \in L} F(x), \quad (3)
\]

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\[
L = \{x \mid Ax \leq a, \; Bx = b, \; x \in X\}.
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If all the functions \(g_m\) are convex, then the max function \(F\) is also a convex function.
The linearly constrained minimax problem

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- Simply a convex NLP problem!
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However, the function $F$ may not be a smooth function.
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However, the function \( F \) may not be a smooth function.

\( \triangleright \) In the ESH subproblem the function \( F \) is rarely a smooth function!
Since $F$ is a nonsmooth function, standard gradient based methods may fail to solve the minimax problem.
Since $F$ is a nonsmooth function, standard gradient based methods may fail to solve the minimax problem. To illustrate this, let’s examine the following example:

$$\text{find } \bar{x} \in \arg\min_{x \in L} F(x),$$

$$F(x) = \max\{g_1(x), g_2(x)\}$$

$$g_1(x) = 0.9x_1^2 + 1.8(x_2 - 5)^2 - 60$$

$$g_2(x) = 0.9x_1^2 + 1.8(x_2 + 5)^2 - 60$$

$$L = \{x \mid -10 \leq x_1 \leq 10, \ -10 \leq x_2 \leq 10\}.$$
The functions $g_1, g_2$ and a contour plot of the max function $F$. 
The functions $g_1$, $g_2$ and a contour plot of the max function $F$.

Suppose we start at the point $x_1 = -6.2, x_2 = 0$
The functions $g_1, g_2$ and a contour plot of the max function $F$.

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At this point $F$ is not differentiable and neither $-\nabla g_1$ or $-\nabla g_2$ gives a descent direction
To solve the minimax problem we can use a nonsmooth method,
To solve the minimax problem we can use a nonsmooth method,
or rewrite it as a smooth problem by adding an auxiliary variable $\mu$

$$\begin{align*}
\text{minimize} & \quad \mu \\
\text{subject to} & \quad g_m(x) \leq \mu \quad \forall m \\
& \quad Ax \leq a \\
& \quad Bx = b \\
& \quad \mu \in \mathbb{R}, x \in X
\end{align*}$$

Resulting in a standard NLP problem.
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\end{align*}
\]

Resulting in a standard NLP problem.

This gave us an idea:

- By utilizing the special problem structure it could possible to solve the problem efficiently by a modified version of Kelley’s cutting plane method.
Basic steps of Kelley’s cutting plane method\textsuperscript{5}

\textsuperscript{5} The cutting-plane method for solving convex programs, Kelley, J. E., Journal of the Society for Industrial & Applied Mathematics 8,703-712 (1960)
Basic steps of Kelley’s cutting plane method\textsuperscript{5}

0. First define $\Omega_0 = \{x, \mu \mid x \in L, \mu_{\text{min}} \leq \mu \leq \mu_{\text{max}}\}$, set $k = 1$ and specify the accepted tolerance $\epsilon$ for the nonlinear constraints.

\textsuperscript{5} The cutting-plane method for solving convex programs, Kelley, J. E., Journal of the Society for Industrial & Applied Mathematics 8,703-712 (1960)
Basic steps of Kelley’s cutting plane method

0. First define $\Omega_0 = \{x, \mu | x \in L, \mu_{\text{min}} \leq \mu \leq \mu_{\text{max}}\}$, set $k = 1$ and specify the accepted tolerance $\epsilon$ for the nonlinear constraints.

Repeat until $g_m(\hat{x}_k) - \mu_k < \epsilon \quad \forall \ m$

1. Find minimum of the objective within the linear set

$$[\hat{x}_k, \mu_k] \in \arg\min_{x, \mu \in \Omega_{k-1}} \mu$$

2. Generate a cutting plane for the most violated constraint at $[\hat{x}_k, \mu_k]$ and update the set $\Omega$

$$l_k(x, \mu) = g_m(\hat{x}_k) + \nabla g_m(\hat{x}_k)^T(x - \hat{x}_k) - \mu$$

$$\Omega_k = \{x, \mu | l_k(x, \mu) \leq 0, \quad x, \mu \in \Omega_{k-1}\}, \quad k = k + 1$$

Since the functions $g_m$ are all convex, all linearizations $g_m(\hat{x}_k) + \nabla g_m(\hat{x}_k)^T (x - \hat{x}_k)$ underestimates the function $g_m$. 

![Graph showing underestimation of a function with a linear approximation.](image)
Since the functions $g_m$ are all convex, all linearizations $g_m(\hat{x}_k) + \nabla g_m(\hat{x}_k)^T(x - \hat{x}_k)$ underestimates the function $g_m$.

Note that the linearizations also underestimates the max function $F$.

---

Since the functions $g_m$ are all convex, all linearizations $g_m(\hat{x}_k) + \nabla g_m(\hat{x}_k)^T(x - \hat{x}_k)$ underestimates the function $g_m$.

Note that the linearizations also underestimates the max function $F$.  

Hence $\mu_k$ gives a lower bound of the optimal value of the maxfunction and an upper bound is given by $F(\hat{x}_k)$. 

The optimality gap is hence given by $F(\hat{x}_k) - \mu_k$ 

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At a point $[\hat{x}_k, \mu_k]$ several of the nonlinear constraints might be violated.

- At such a point it is possible to generate several cutting planes.
- How to choose which cutting planes?
At a point $[\hat{x}_k, \mu_k]$ several of the nonlinear constraints might be violated.

- At such a point it is possible to generate several cutting planes.
- How to choose which cutting planes?
- Functions such that $g_m(\hat{x}_k) = F(\hat{x}_k)$ describes the function $F$ within the neighborhood of $\hat{x}_k$. 
Kelley’s cutting plane method often takes too long steps and “jumps” over the optimal solution.
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Let’s consider the following example:

minimize $F(x) = x^2$, $x \in [-2, 2]$
Kelley’s cutting plane method often takes to long steps and "jumps" over the optimal solution.

Let's consider the following example:

minimize $F(x) = x^2$, $x \in [-2, 2]$

minimize $\mu$

subject to $x^2 \leq \mu$

$-2 \leq x \leq 2$

$-10 \leq \mu \leq 10$

$x, \mu \in \mathbb{R}$

(4)
Kelley’s cutting plane method

\[
\begin{align*}
\hat{x}_1, \mu_1 \\
\hat{x}_2, \mu_2 \\
\hat{x}_3, \mu_3 \\
\hat{x}_4, \mu_4 \\
\hat{x}_5, \mu_5
\end{align*}
\]
Kelley’s cutting plane method

\[
\begin{align*}
\hat{x}_1, \mu_1
\end{align*}
\]
Kelley’s cutting plane method

\[ \hat{x}_1, \mu_1 \]
Kelley’s cutting plane method

\[
\begin{align*}
\hat{x}_1 - 3 \hat{x}_2 - 2 \hat{x}_3 - 1 &= -10 \quad (\hat{x}_1, \mu_1) \\
-3 \hat{x}_2 - 2 \hat{x}_3 - 1 &= -5 \quad (\hat{x}_2, \mu_2)
\end{align*}
\]
Kelley's cutting plane method

\[
\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5, \mu_1, \mu_2
\]
Kelley’s cutting plane method

\[
\hat{x}_1, \mu_1 \quad \hat{x}_2, \mu_2 \quad \hat{x}_3, \mu_3
\]
Kelley’s cutting plane method
Kelley’s cutting plane method

\[
\begin{align*}
\hat{x}_1 & \approx 1 \\
\hat{x}_2 & \approx -3 \\
\hat{x}_3 & \approx 5 \\
\hat{x}_4 & \approx 10 \\
\hat{x}_5 & \\
\end{align*}
\]

\[
\begin{align*}
\mu_1 & \\
\mu_2 & \\
\mu_3 & \\
\mu_4 & \\
\mu_5 & \\
\end{align*}
\]
Kelley’s cutting plane method

\[ \left( \hat{x}_1, \mu_1 \right), \left( \hat{x}_2, \mu_2 \right), \left( \hat{x}_3, \mu_3 \right), \left( \hat{x}_4, \mu_4 \right) \]
Kelley’s cutting plane method

\[
\hat{x}_4, \mu_4 \ (\hat{x}_5, \mu_5) \\
\hat{x}_2, \mu_2 \ (\hat{x}_1, \mu_1) \\
\hat{x}_3, \mu_3
\]
However, by taking advantage of the problem structure we can stop in the 4:th iteration and verify that the optimal solution was found already in iteration 3

\[ F(x_3) - \mu_4 = 0. \]
In case cutting planes are generated for all function $g_m$ such that $g_m(\hat{x}_{k-1}) = F(\hat{x}_{k-1})$ and $\hat{x}_{k-1}$ is not the optimal solution, it can easily be shown $(\hat{x}_k - \hat{x}_{k-1})$ gives a descent direction for the max function at $\hat{x}_{k-1}$. 
In case cutting planes are generated for all function $g_m$ such that $g_m(\hat{x}_{k-1}) = F(\hat{x}_{k-1})$ and $\hat{x}_{k-1}$ is not the optimal solution, it can easily be shown $(\hat{x}_k - \hat{x}_{k-1})$ gives a descent direction for the max function at $\hat{x}_{k-1}$.

Since both $\hat{x}_k$ and $\hat{x}_{k-1}$ fulfills all constraints of the minimax problem we can preform a line search between these points for the minimum of the max function $F$,

\[
\lambda = \arg\min_{\lambda \in [0,1]} F(\lambda \hat{x}_k + (1 - \lambda)x_{k-1}).
\]  

If $\lambda \neq 1$ a better solution is obtained at $x_k = \lambda \hat{x}_k + (1 - \lambda)\hat{x}_{k-1}$.

A new upper bound for the objective is given by $F(x_k)$ and additional cutting planes can be generated at $x_k$. 
Basic steps of the modified method
Basic steps of the modified method

0. First define \( \Omega_0 = \{ x, \mu | x \in L, \mu_{\text{min}} \leq \mu \leq \mu_{\text{max}} \} \), set \( k = 1 \) and specify the accepted optimality gap \( \epsilon \)
Basic steps of the modified method

0. First define $\Omega_0 = \{x, \mu | x \in L, \mu_{\text{min}} \leq \mu \leq \mu_{\text{max}}\}$, set $k = 1$ and specify the accepted optimality gap $\epsilon$

Repeat until $F(x_k) - \mu_k < \epsilon$

1. Find minimum of the objective within the linear set

$$[\hat{x}_k, \mu_k] \in \arg\min_{x, \mu \in \Omega_{k-1}} \mu$$

2. If $k > 1$,

$$\text{find } \lambda = \arg\min_{\lambda \in [0,1]} F(\lambda \hat{x}_k + (1 - \lambda)\hat{x}_{k-1})$$

and set $x_k = \lambda \hat{x}_k + (1 - \lambda)\hat{x}_{k-1}$.

3. Generate cutting planes $l_k(x, \mu)$ for all functions $g_m$ such that $g_m(x) = F(x)$ at the both $x_k$ and $\hat{x}_k$ and update the set $\Omega$

$$\Omega_k = \{x, \mu | l_k(x, \mu) \leq 0, \quad x, \mu \in \Omega_{k-1}\} \quad k = k + 1$$
The method was implemented in Matlab 2013.

- Gurobi 6.0.3 was used as a subsolver for the LP subproblems.
- For the line search we used Matlab’s fminbnd.
- Unconstrained variables are given limits of \(-10^{20}\) and \(10^{20}\).
The method was implemented in Matlab 2013.

- Gurobi 6.0.3 was used as a subsolver for the LP subproblems.
- For the line search we used Matlab’s fminbnd.
- Unconstrained variables are given limits of $-10^{20}$ and $10^{20}$.

As a test set we have used all MINLP problems classified as convex and containing at least two nonlinear constraints within MINLPlib 2.

- These have been rewritten as nonlinear minimax problems, where the objective is to minimize the pointwise maximum of the nonlinear constraint functions and obtain an interior point.
The method was implemented in Matlab 2013.
  ▶ Gurobi 6.0.3 was used as a subsolver for the LP subproblems.
  ▶ For the line search we used Matlab’s fminbnd.
  ▶ Unconstrained variables are given limits of $-10^{20}$ and $10^{20}$.

As a test set we have used all MINLP problems classified as convex and containing at least two nonlinear constraints within MINLPlib 2.
  ▶ These have been rewritten as nonlinear minimax problems, where the objective is to minimize the pointwise maximum of the nonlinear constraint functions and obtain an interior point.

To evaluate the performance we have compared the Matlab implementation against the following NLP solvers in GAMS: CONOPT, IPOPT, IPOPTH, SNOPT and MINOS.
Modified cutting plane method

Solution time (s)

Minimax problems solved within 0.1% of best found solution

IPOPTH(193)
Modified cutting plane method

Minimax problems solved within 0.1% of best found solution

- IPOPTH(193)
- Kelley’s mod.(192)
Modified cutting plane method

Minimax problems solved within 0.1% of best found solution

IPOPTH (193)  Kelley’s mod. (192)  IPOPT (191)
Minimax problems solved within 0.1% of best found solution

- IPOPTH (193)
- Kelley’s mod. (192)
- IPOPT (191)
- CONOPT (188)
Minimax problems solved within 0.1% of best found solution

- IPOPTH (193)
- Kelley’s mod. (192)
- IPOPT (191)
- CONOPT (188)
- Kelley’s (169)
Minimax problems solved within 0.1% of best found solution

Solution time (s)

IPOPTH (193)  Kelley’s mod. (192)
IPOPT (191)    CONOPT (188)
Kelley’s (169) SNOPT (115)
Modified cutting plane method

Minimax problems solved within 0.1% of best found solution

- IPOPTH (193)
- Kelley’s mod. (192)
- IPOPT (191)
- CONOPT (188)
- Kelley’s (169)
- SNOPT (115)
- MINOS (95)

Solution time (s)
Minimax problems solved within 0.01% of best found solution

- IPOPTH (172)
- Kelley’s mod. (175)
- IPOPT (170)
- CONOPT (188)
- Kelley’s (155)
- SNOPT (108)
- MINOS (87)
Modified cutting plane method

How does the modifications affect the performance?

Minimax problems solved within 0.1% of best found solution

Kelley’s (169)
Modified cutting plane method

How does the modifications affect the performance?

Solution time (s)

Minimax problems solved within 0.1% of best found solution

Kelley’s (169) + termination crit. (170)
Modified cutting plane method

How does the modifications affect the performance?

Minimax problems solved within 0.1% of best found solution

- Kelley’s (169)
- +termination crit. (170)
- +additional cuts (189)
How does the modifications affect the performance?

![Graph showing solution time vs. Minimax problems solved within 0.1% of best found solution. The graph compares Kelley’s method, Kelley’s method with additional cuts, Kelley’s method with termination criteria, and Kelley’s method with line search. Each line represents the performance of the respective method over solution time in seconds.]}
However, in the ESH algorithm we do not need an optimal solution of the minimax problems. The solution $\bar{x}$ only has to be within the set $C$, i.e., $F(\bar{x}) < 0$. 
However, in the ESH algorithm we do not need an optimal solution of the minimax problems. The solution $\bar{x}$ only has to be within the set $C$, i.e., $F(\bar{x}) < 0$.

Such a solution can be found much faster.
How to efficiently find an interior point of the set $C = \{ x \mid g_m(x) \leq 0 \quad \forall m, x \in \mathbb{R}^n \}$, when the functions $g_m$ are pseudo convex?

Rewriting the problem results in the following nonconvex constraints: $g_m(x) - \mu \leq 0$. 
How to efficiently find an interior point of the set $C = \{x \mid g_m(x) \leq 0 \quad \forall m, x \in \mathbb{R}^n\}$, when the functions $g_m$ are pseudo convex?

- Rewriting the problem results in the following nonconvex constraints: $g_m(x) - \mu \leq 0$. 

![Graph](image)
Future work

- How to efficiently find an interior point of the set
  \[ C = \{ x \mid g_m(x) \leq 0 \quad \forall m, x \in \mathbb{R}^n \} \], when the functions \( g_m \) are pseudo convex?
  
  - Rewriting the problem results in the following nonconvex constraints:
    \[ g_m(x) - \mu \leq 0. \]

- These minimax problems can be solved with a method similar to \( \alpha \)ECP, which is a method for solving pseudoconvex MINLP problems.\(^7\)

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\(^7\) Solving pseudo-convex mixed integer optimization problems by cutting plane techniques, Westerlund, T., Pörn, R., Optimization and Engineering 3, 253-280 (2002).
Thank you for your attention!
Any questions?