A SHOT AT CONVEX OPTIMIZATION

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ose OPTIMIZATION AND SYSTEMS ENGINEERING

Åbo Akademi University

CONTENTS OF THE TALK

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Results from an **extensive benchmark of SHOT** against several other MINLP solvers is provided.

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The ESH algorithm and the SHOT solver is described in:

> *Kronqvist J., Lundell A. and Westerlund T., The extended supporting hyperplane algorithm for convex MINLP problems, Journal of Global Optimization, accepted 2015*

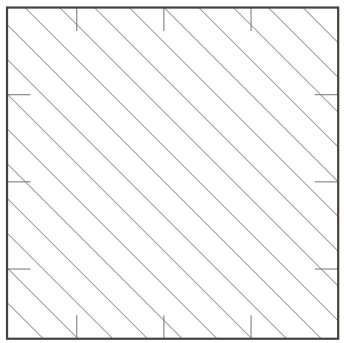
THE MINLP PROBLEM SCOPE

The ESH algorithm solves convex MINLP problems of the type

$$\text{find } x^* \in \arg\min_{x \in C \cap L \cap Y} c^T x \qquad \text{(P)}$$

where $x = [x_1, x_2, \ldots, x_N]^T$ belongs to the compact set

$$X = \{x \mid \underline{x}_i \leq x_i \leq \overline{x}_i,\ i = 1, \ldots, N\} \subset \mathbb{R}^n,$$



the feasible region is defined by $C \cap L \cap Y$

$$\begin{aligned} C &= \{x \mid g_m(x) \leq 0,\ m = 1, \ldots, M,\ x \in X\} \\ L &= \{x \mid Ax \leq a,\ Bx = b,\ x \in X\} \\ Y &= \{x \mid x_i \in \mathbb{Z},\ i \in I_{\mathbb{Z}},\ x \in X\} \end{aligned}$$

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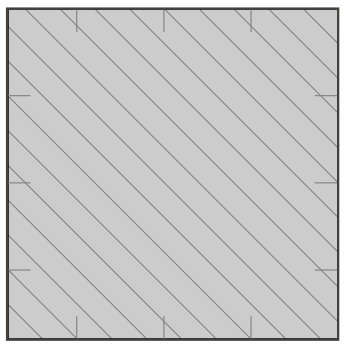

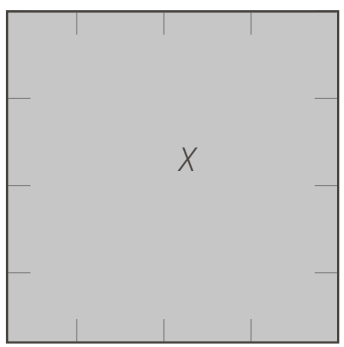

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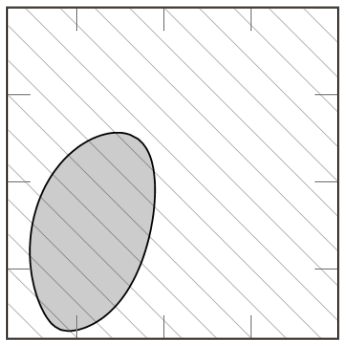

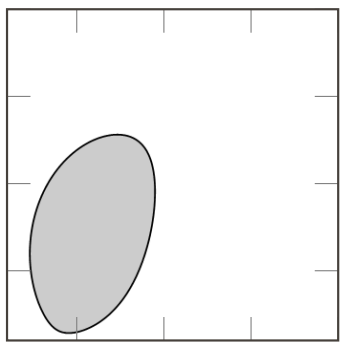

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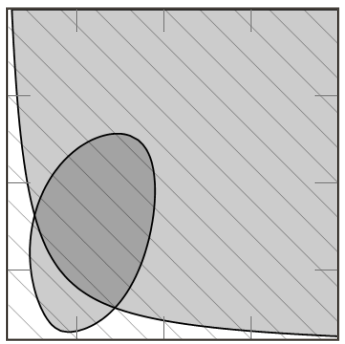

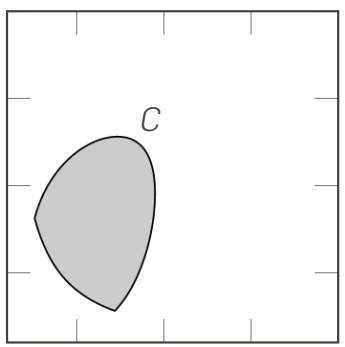

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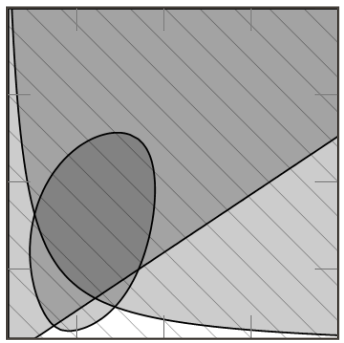

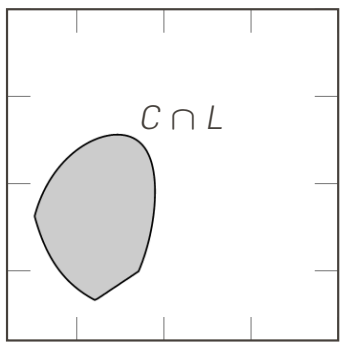

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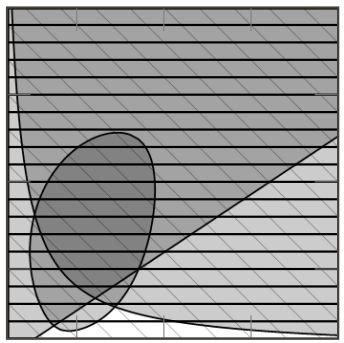

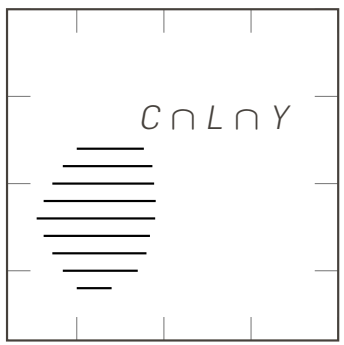

BREAKDOWN OF THE ESH ALGORITHM

Interior point search step

Obtain a feasible, relaxed interior point (satisfying C) by solving a NLP problem.

LP step (optional)

Solve simple LP problems (initially in $X \cap L$) to obtain an initial overestimating linear set.

MILP step

Solve MILP problems to find the optimal solution to (P).

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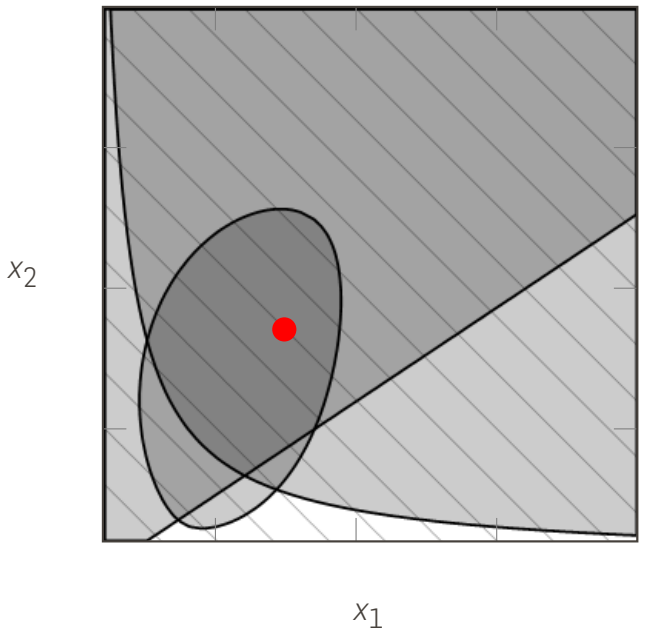
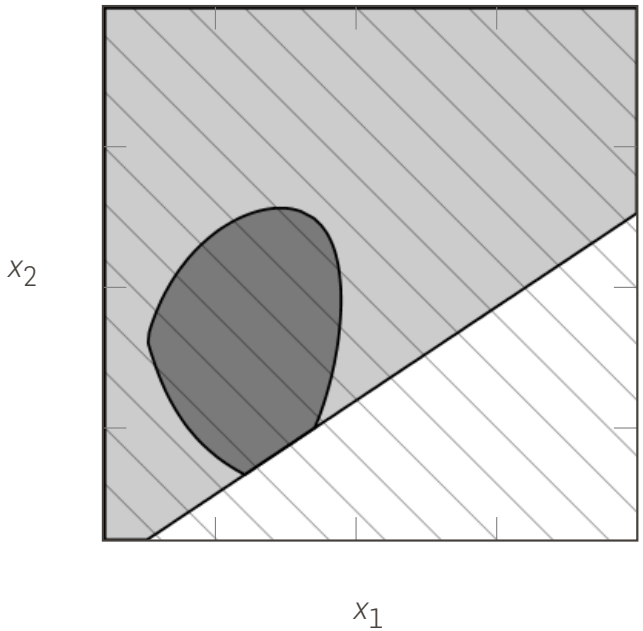


If an interior point is not given, obtain a feasible, relaxed interior point (satisfying all the nonlinear constraints in C) by solving a NLP problem.

LP STEP (OPTIONAL)



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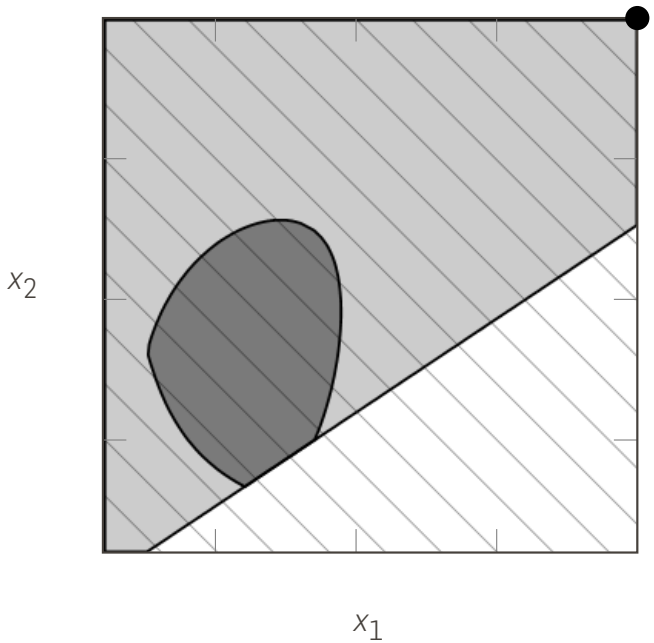

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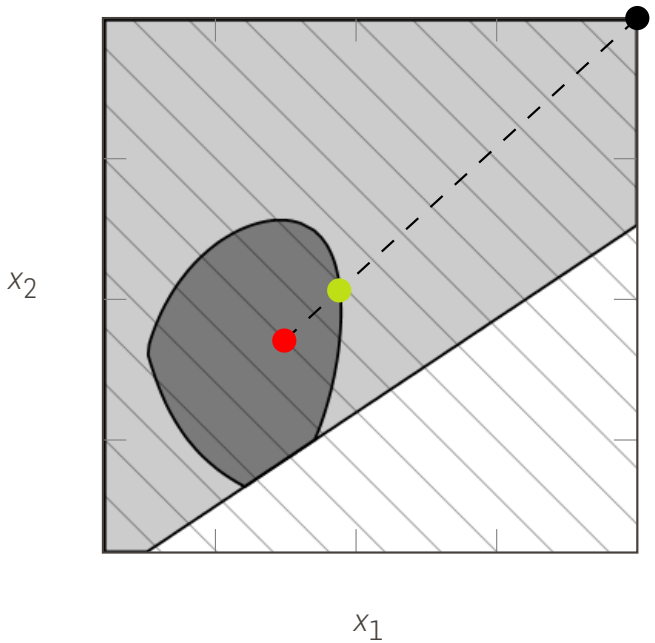

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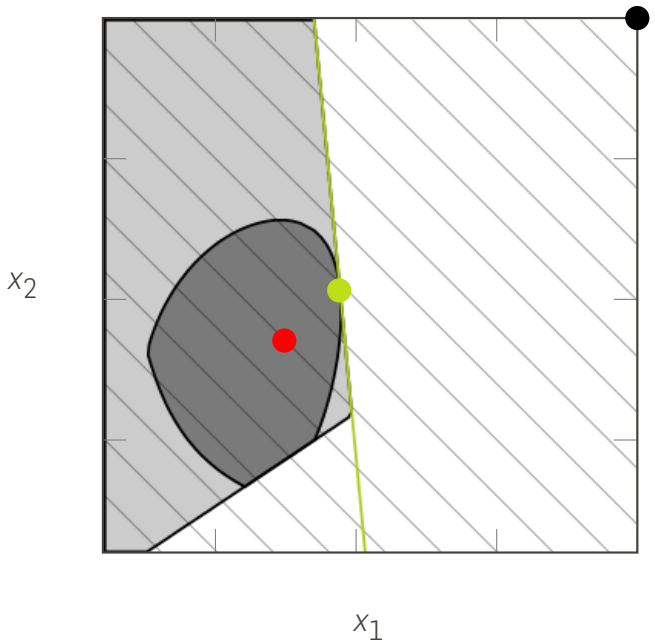

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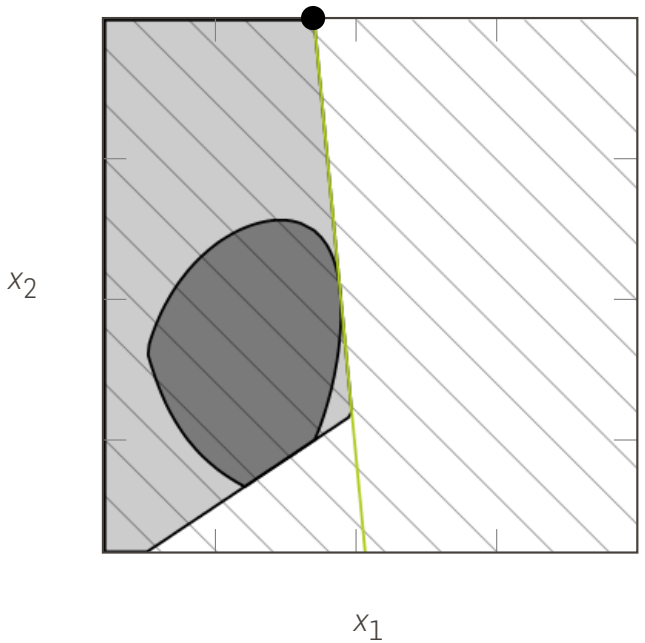

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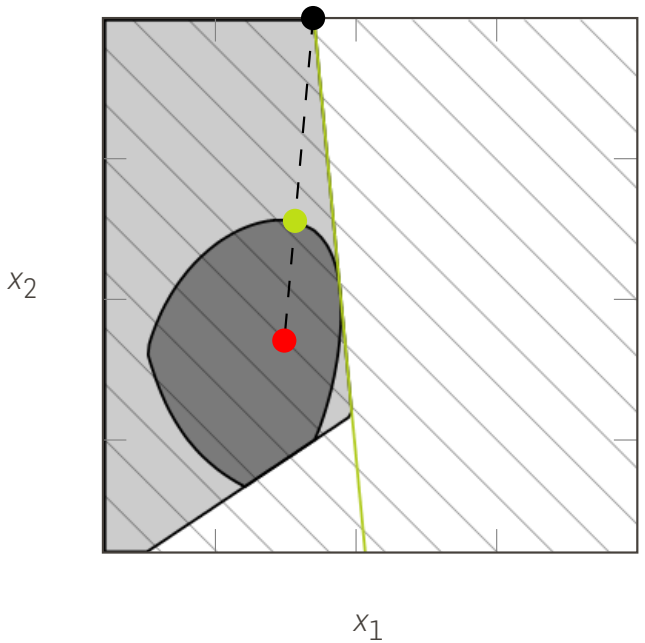

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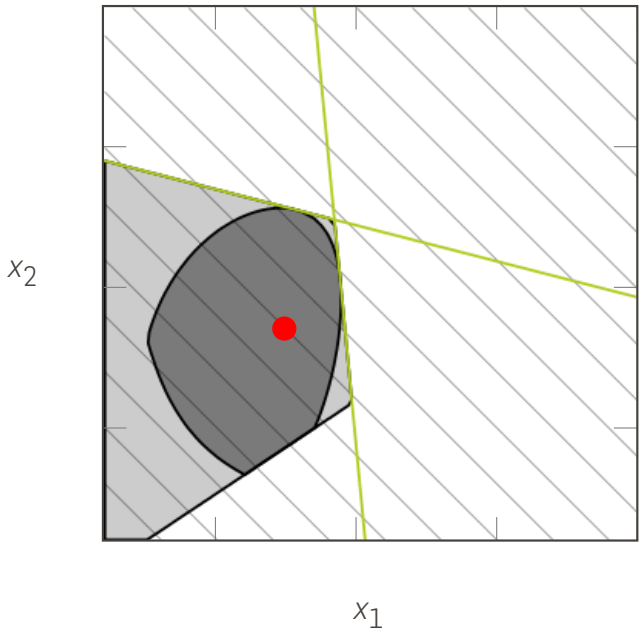

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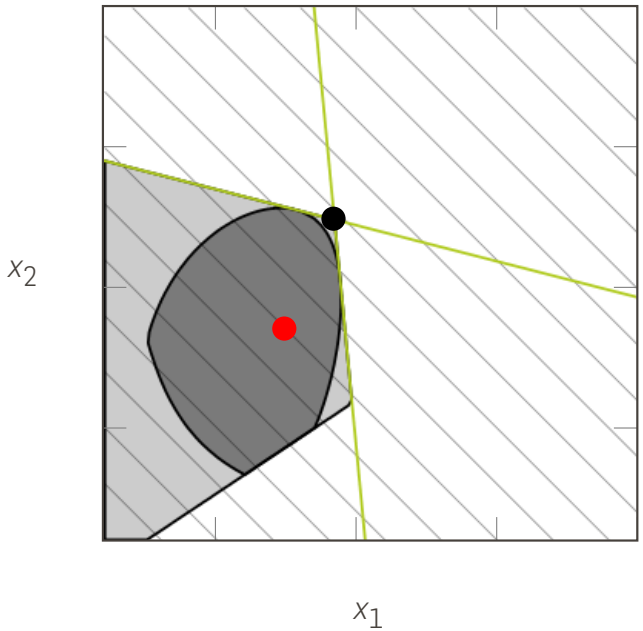

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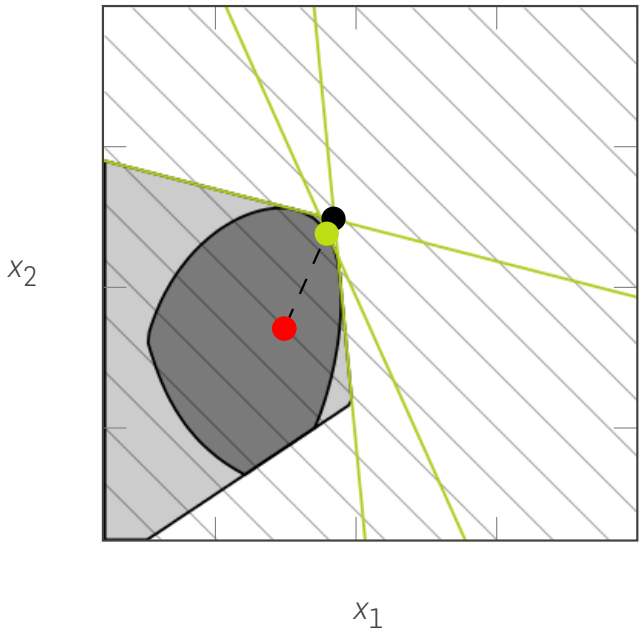

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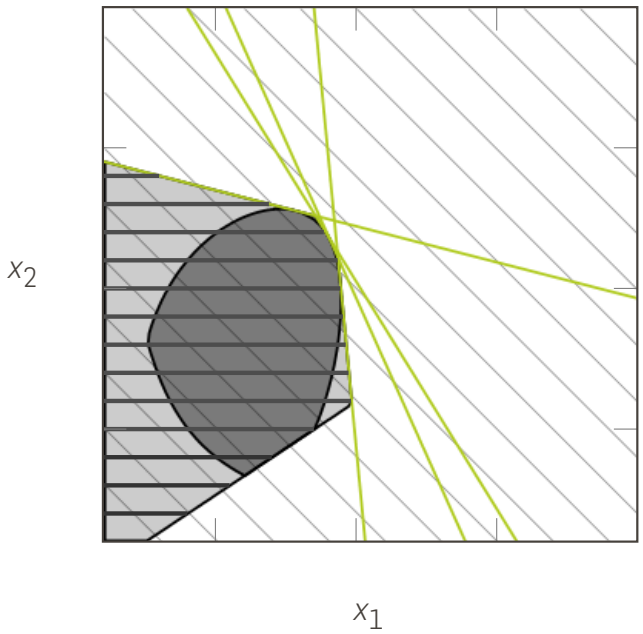

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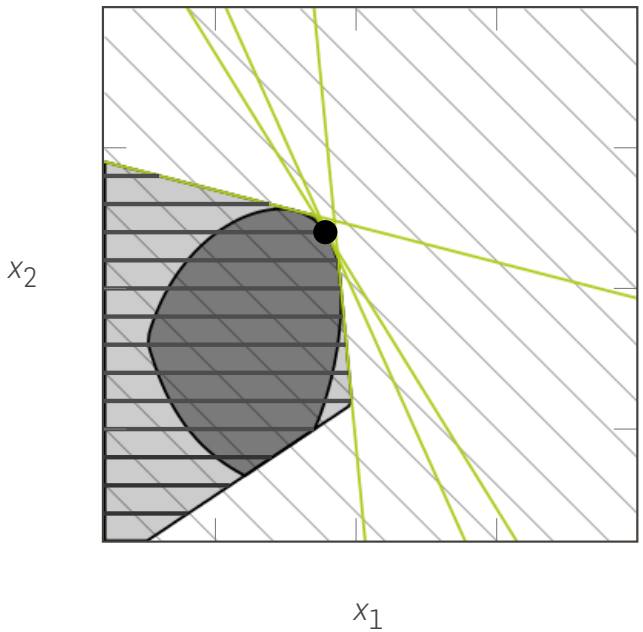

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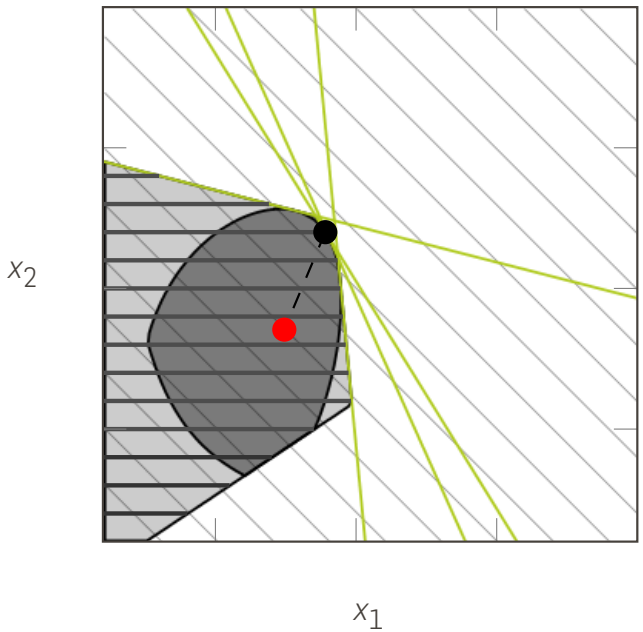

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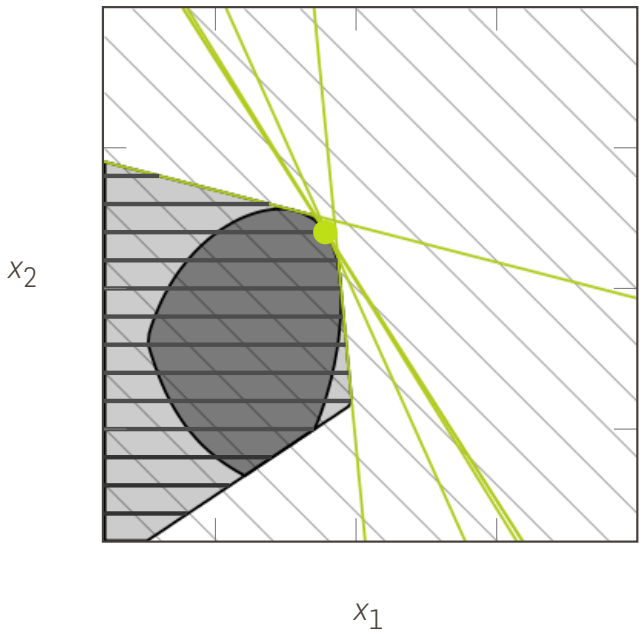

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SHOT will be released as an open source solver in COIN-OR.

FILE FORMATS SUPPORTED

SHOT reads problems in OSiL (Optimization Services instance Language) format and NL format.

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  <variables numberOfVariables="8">
    <var name="x1" />
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    <var name="b7" type="B" ub="1" />
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  </objectives>
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    <con name="e4" ub="0" />
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    </start>
    <colIdx>
      <el mult="4" incr="1">0</el>
      <el mult="4" incr="1">0</el>
      <el>0</el>
```

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Options are specified in OSoL (Optimization Services options Language) format.

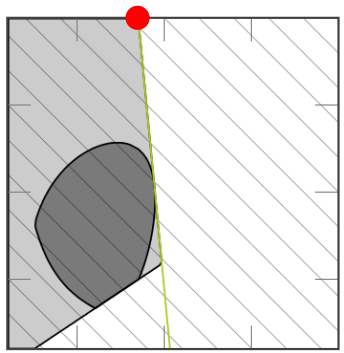
Results are provided in OSrL (Optimization Services results Language) format.

OSiL, OSoL and OSrL are XML-based and part of the Optimization Services project.

DUAL AND PRIMAL SOLUTIONS

A dual solution provides the best known lower bound on the objective value:

- Belongs to the relaxed set Ω_k.
- Provided by optimal solutions to the LP/MILP subproblems.



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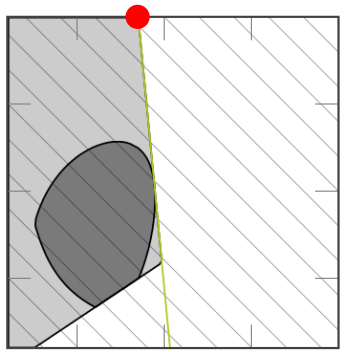
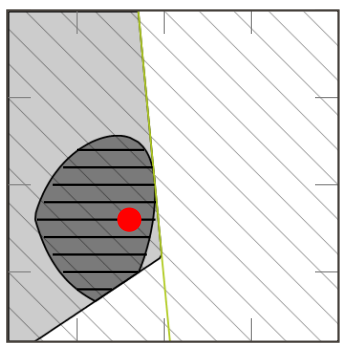


A primal solution provides the best known integer-feasible solution to the MINLP problem:

- Belongs to the feasible set $C \cap L \cap Y$.
- Provided by primal heuristics.



DUAL SOLUTIONS (LOWER BOUND)

The dual solutions to the MINLP problem is given as the solutions to the LP/MILP subproblems.

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  - LP solutions valid if found
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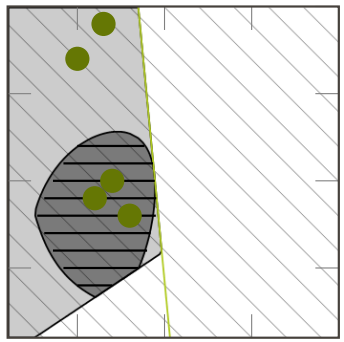
Many MILP solvers support a solution pool, *i.e.*, can return several feasible solutions

- these can be used to create more hyperplanes in each iteration



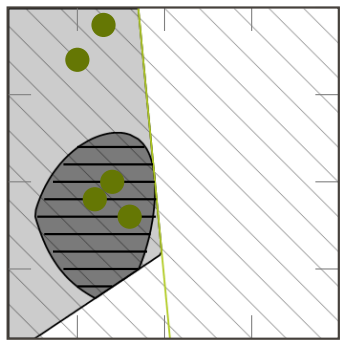
DUAL SOLUTIONS (LOWER BOUND)

The dual solutions to the MINLP problem is given as the solutions to the LP/MILP subproblems.

- Only valid if the problems are solved to optimality
  - LP solutions valid if found
  - MILP solutions valid only if flagged optimal by the subsolver

Many MILP solvers support a solution pool, *i.e.*, can return several feasible solutions

- these can be used to create more hyperplanes in each iteration



BUT… Too many hyperplanes may make each subsequent iteration computationally more expensive.

PRIMAL SOLUTIONS (UPPER BOUND)

Several techniques are used to find primal solutions:

- If points in the MILP solution pool are also in C these are primal solutions.
- Fix the integer-variables to the integer-valid solution and solve an NLP problem.

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Future work

- Include more primal heuristics in SHOT, *e.g.*, based on line searches.

PRIMAL SOLUTIONS FROM THE MILP SOLUTION POOL

Several parameters available in the MILP solver that determine the returned solutions in the solution pool

`MIPEmphasis`, `Probe`, `SolnPoolGap`, `SolnPoolIntensity`…

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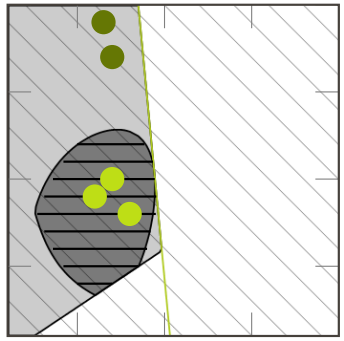
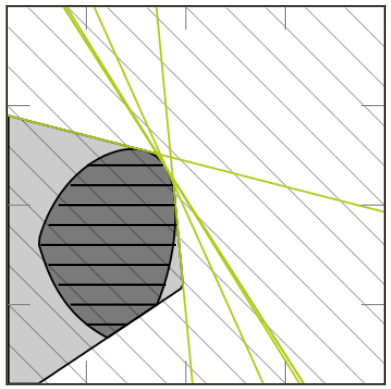
Often it is better to obtain points with ‘bad’ objective values in the linear relaxation for the primal solutions.



PRIMAL SOLUTIONS FROM SOLVING NLP PROBLEMS

To obtain a primal solution, it is possible to fix the integer-values of an integer-feasible solution (not necessarily in $C \cap L$) and solve an NLP problem.



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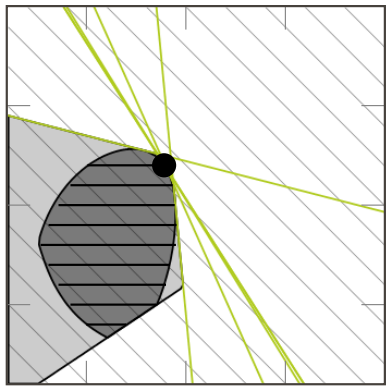

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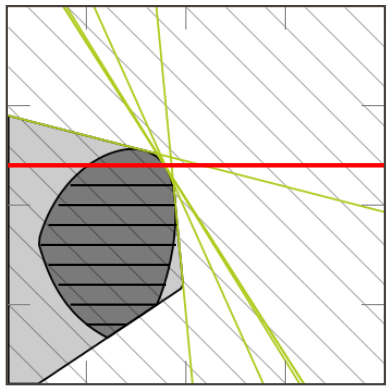

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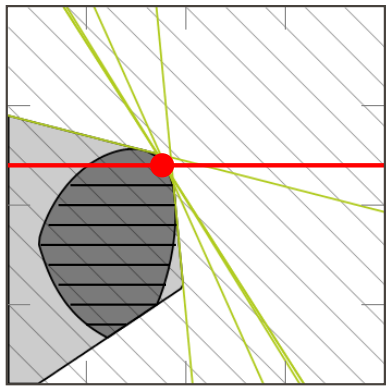

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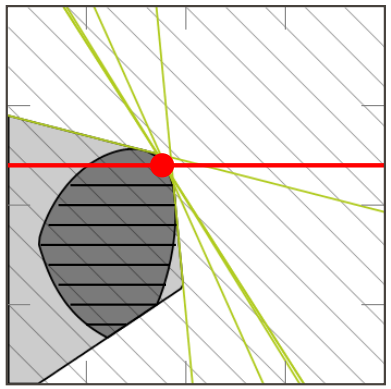

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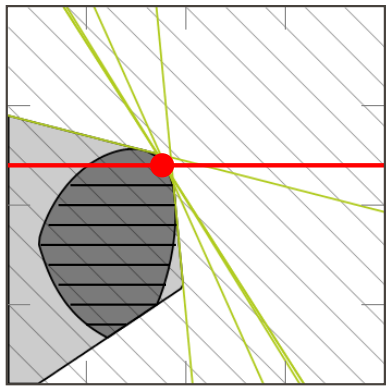

The NLP problem is not always feasible!

In SHOT, Ipopt is used to solve the NLP problem:

- Strategy executed at a specific iteration or time interval
- If the solution is on the boundary of the feasible region or outside (to an ϵ tolerance), a supporting hyperplane is generated.

TERMINATION CRITERIA IN SHOT

Absolute or relative objective duality gap

$$|\mathrm{DB} - \mathrm{PB}| \leq \epsilon_{\mathrm{abs}} \qquad \frac{|\mathrm{DB} - \mathrm{PB}|}{10^{-10} + |\mathrm{PB}|} \leq \epsilon_{\mathrm{rel}}$$

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Iteration or time limit reached

QUADRATIC AND NONLINEAR OBJECTIVE FUNCTIONS

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Quadratic constraints can also be handled by some solvers.

- Numerical issues may lead to trouble with proving semidefiniteness.
- Not always more effective than using the ESH algorithm for these constraints.

PERFORMANCE OF SHOT

SHOT was tested on all 333 MINLP instances classified as convex in the MINLP Library 2:

- Number of variables in the problems 3 – 107 223 (mean 999).
- Largest number of discrete variables in a problem is 1500.
- All benchmarks performed on Linux-based 64 bit computer (Intel Xeon 3.6 GHz, four physical and eight logical cores) with 32 GB RAM. Subsolvers used were CPLEX 12.6.1 and IPOPT 3.11.7.

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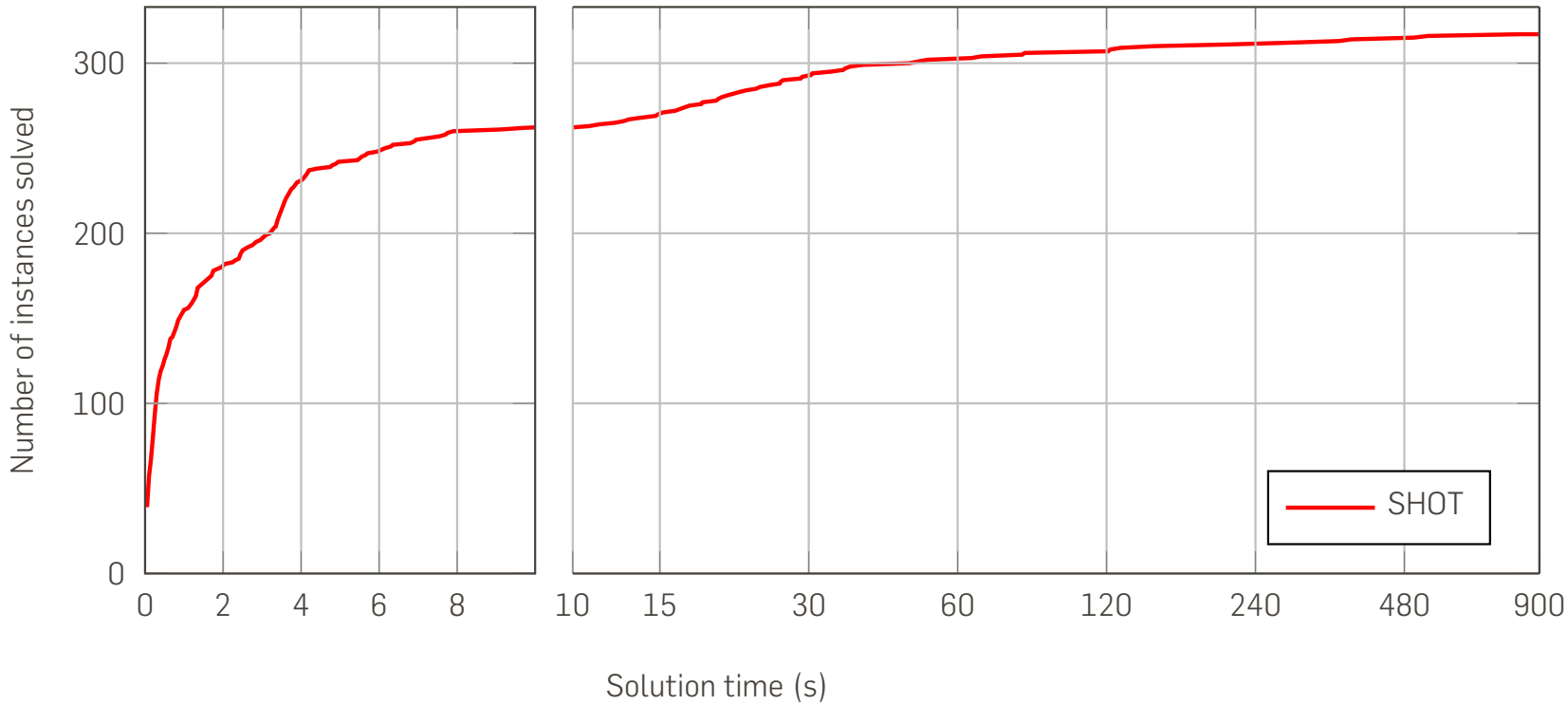
Solution strategy

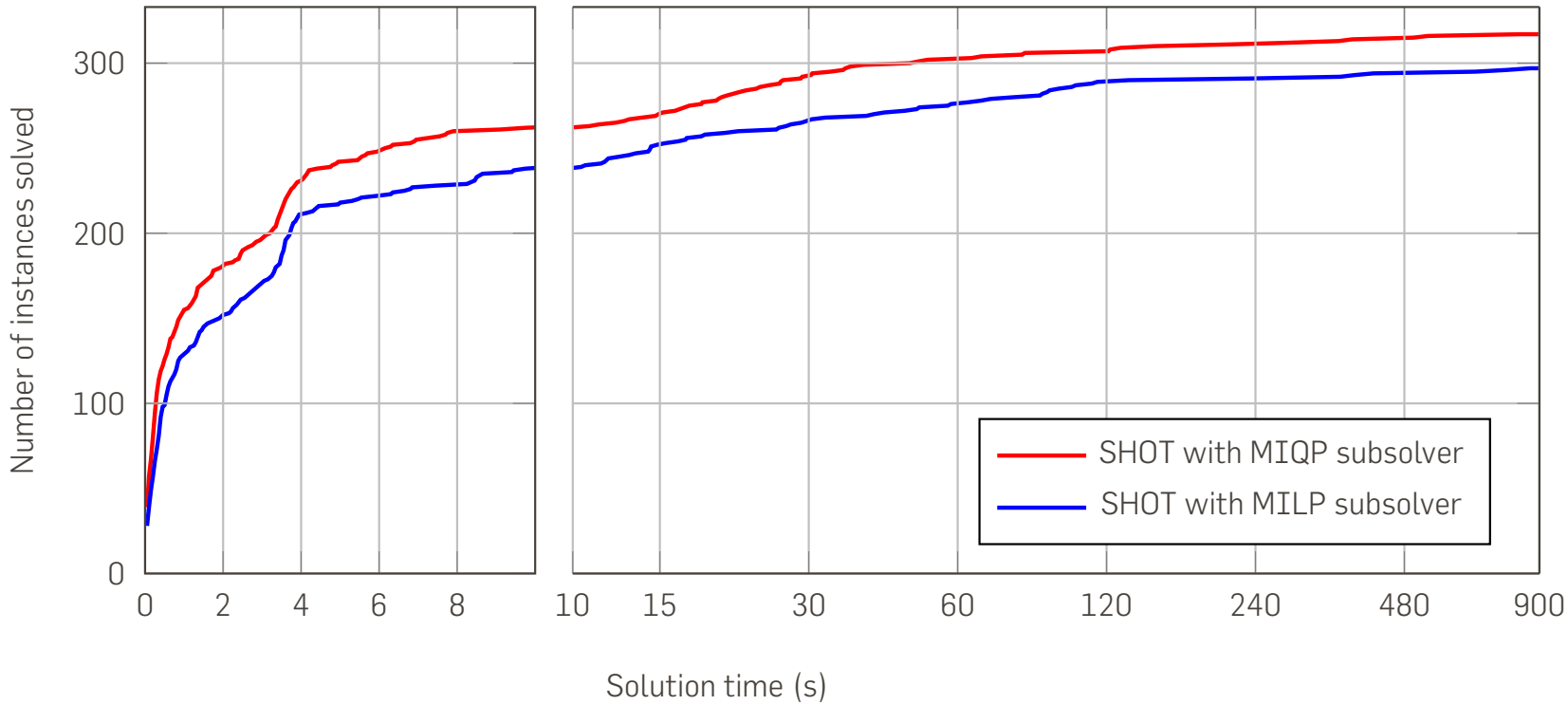
- $\epsilon_{\text{abs}} = \epsilon_{\text{rel}} = 0.001$, $\epsilon_{\text{MILP}} = 10^{-5}$, $\epsilon_{\text{LP}} = 0.001$, $K_{\text{LP}} = 300$
- maximal solution pool size: 10
- quadratic objective functions passed on to subsolver
- quadratic constraints regarded as general nonlinear

PERFORMANCE OF SHOT



A performance profile of the number of problem instances solved by SHOT to an objective duality gap $\leq 1\%$ as calculated by PAVER 2.

IMPACT OF USING THE QUADRATIC OBJECTIVE STRATEGY



A performance profile of the number of problem instances solved by SHOT to an objective duality gap $\leq 1\%$ with a MIQP or MILP subsolver.

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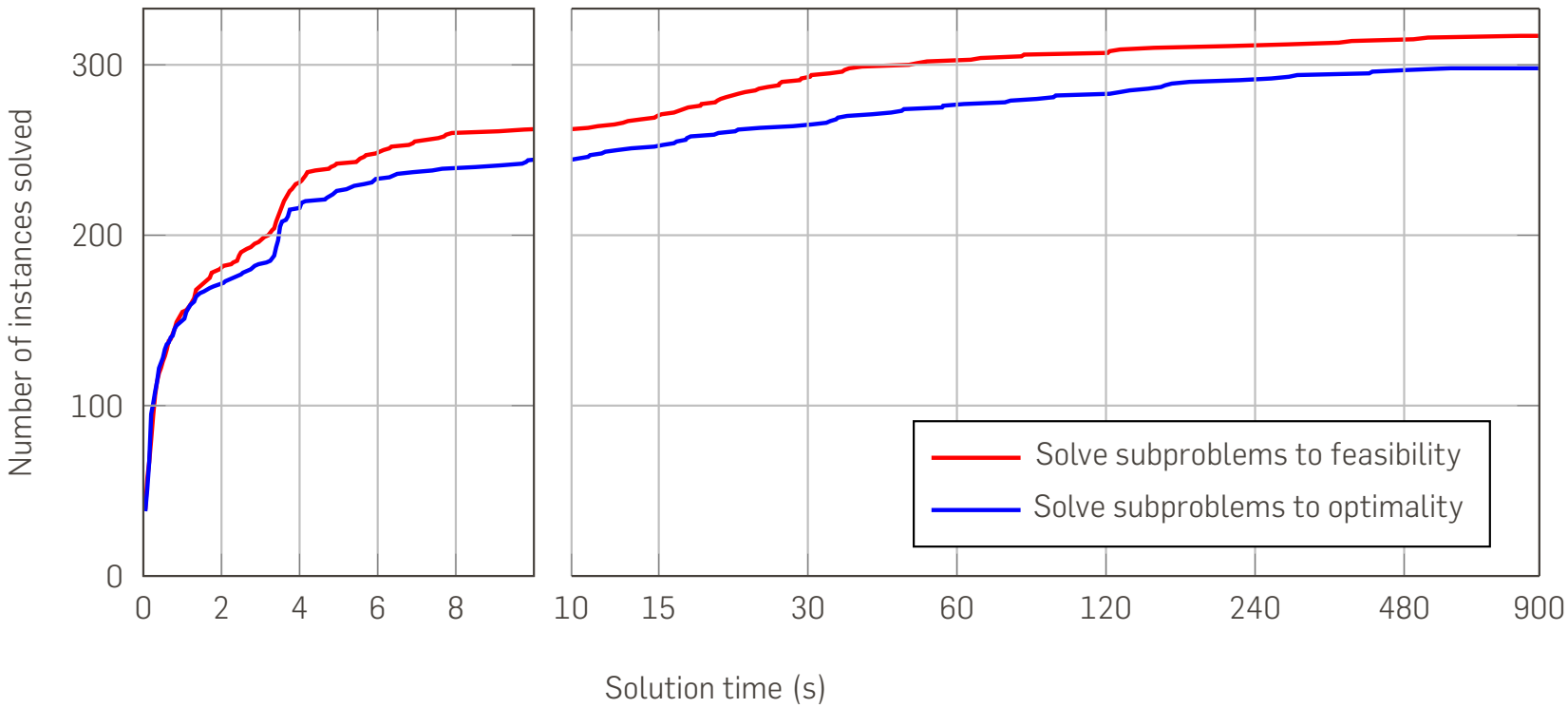
Solution limit strategy

1. Initially set SOLLIM $= 1$.
2. Solve MILP/MIQP subproblem and obtain solution x.
3. Terminate if x is MILP optimal and ESH termination criterion fulfilled.
4. Increase SOLLIM and goto step 2 if x is MILP optimal and $\max_m g_m(x) \leq \epsilon_{\text{SL}}$.
5. Add supporting hyperplanes and goto step 2.

SOLUTION LIMIT STRATEGY IMPACT



Performance profiles of the number of problem instances solved by SHOT to an objective duality gap $\leq 1\%$ with or without an increasing solution limit strategy.

LP RELAXATION STRATEGY

Initially integer-relaxed MILP/MIQP problems, *i.e.*, LP/QP problems, can be solved:

+ Integer-relaxed problems are much faster to solve.
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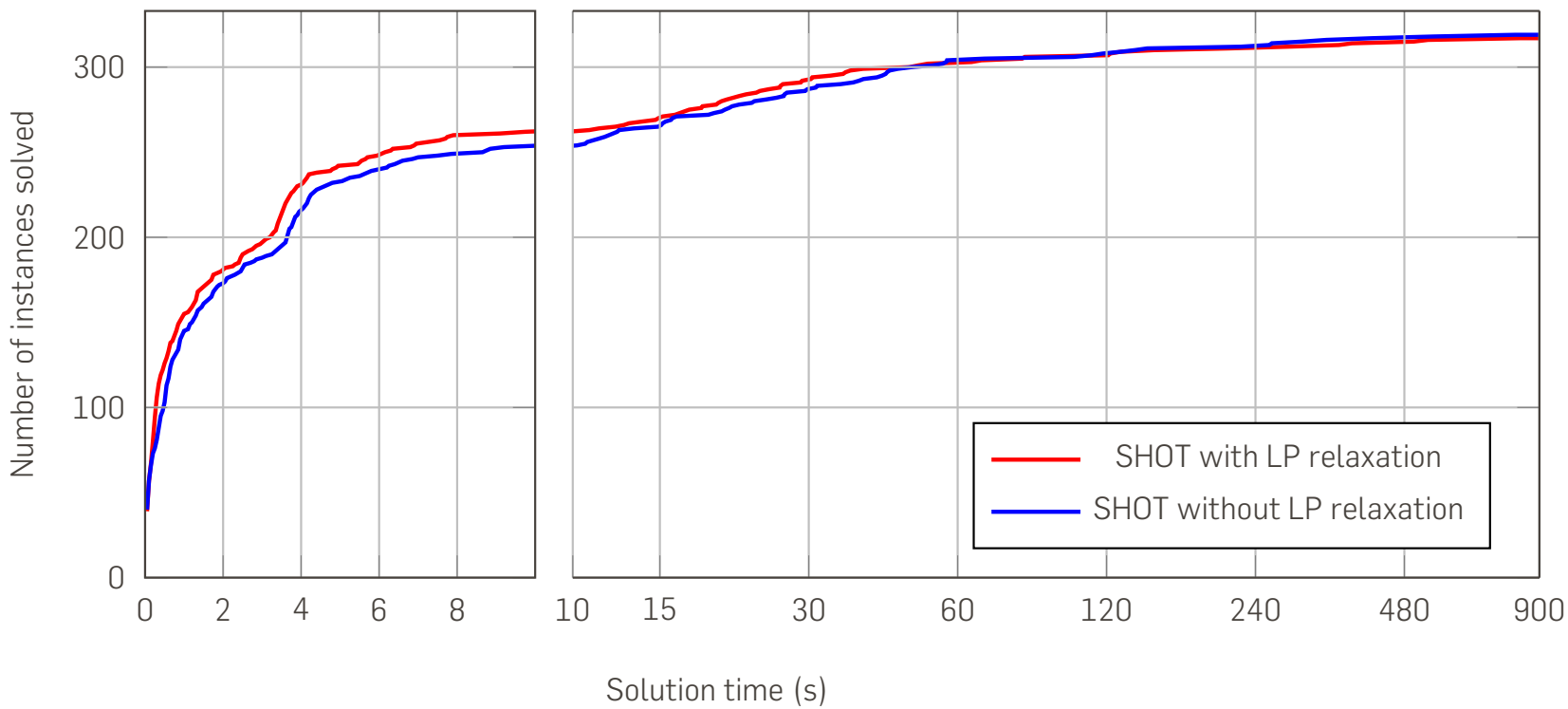
Future work

» Investigate adding these supporting hyperplanes as lazy constraints.

LP RELAXATION STRATEGY IMPACT



Performance profiles of the number of problem instances solved by SHOT to an objective gap $\leq 1\%$ with or without a strategy for solving integer-relaxed problems.

BENCHMARKS AGAINST OTHER MINLP SOLVERS

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The following MINLP solvers available in GAMS 24.4.1 were used:

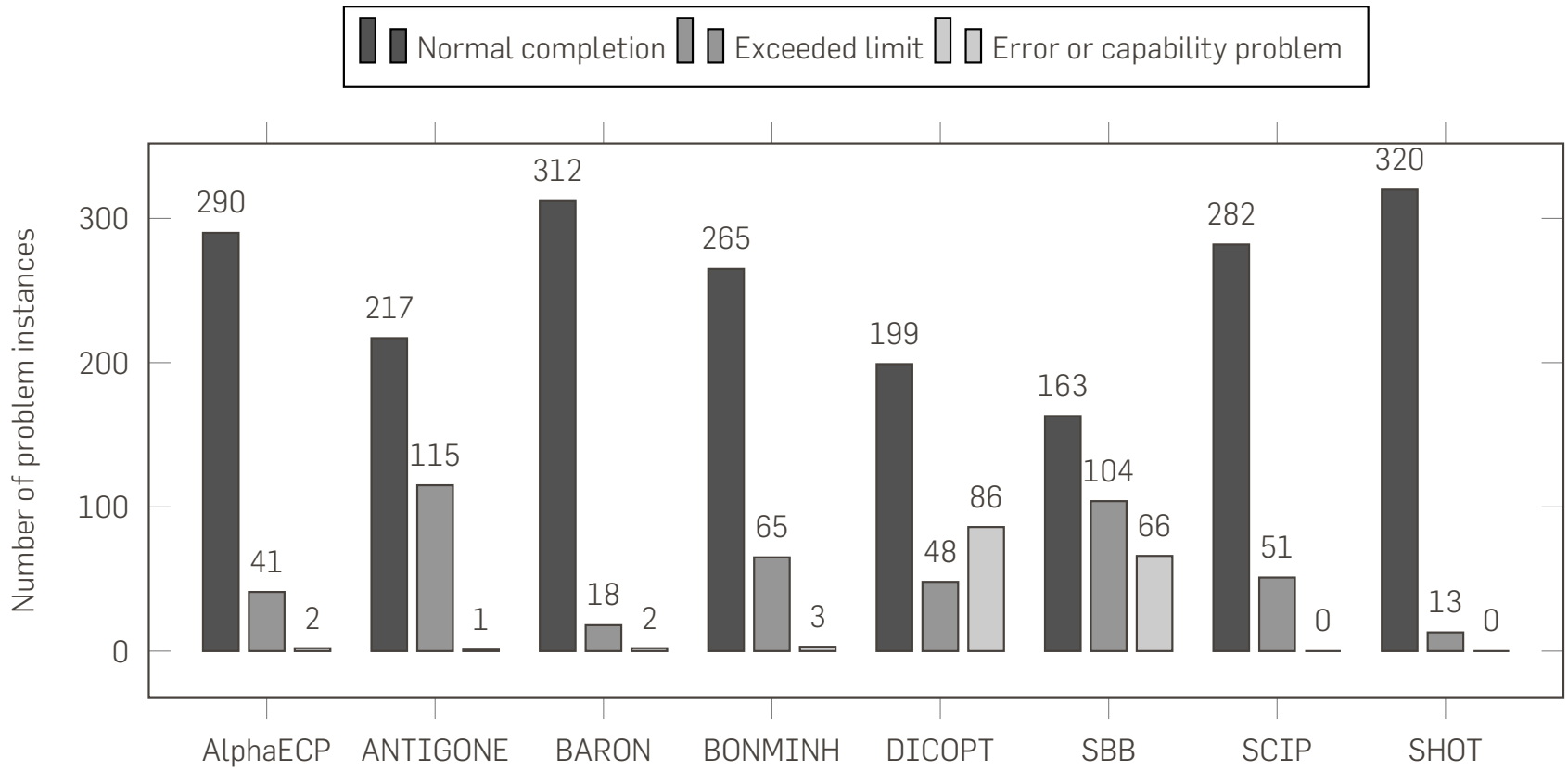
- AlphaECP (with convex strategy)
- ANTIGONE
- BARON
- BONMINH (with recommended convex strategy, B-Hyb)
- DICOPT
- SBB
- SCIP (with convex strategy)

CPLEX and CONOPT were used as subsolvers and the time limit per problem was 900 s.

The same problem set as before was used, *i.e.*, all 333 convex MINLP problems in MINLPLib 2.

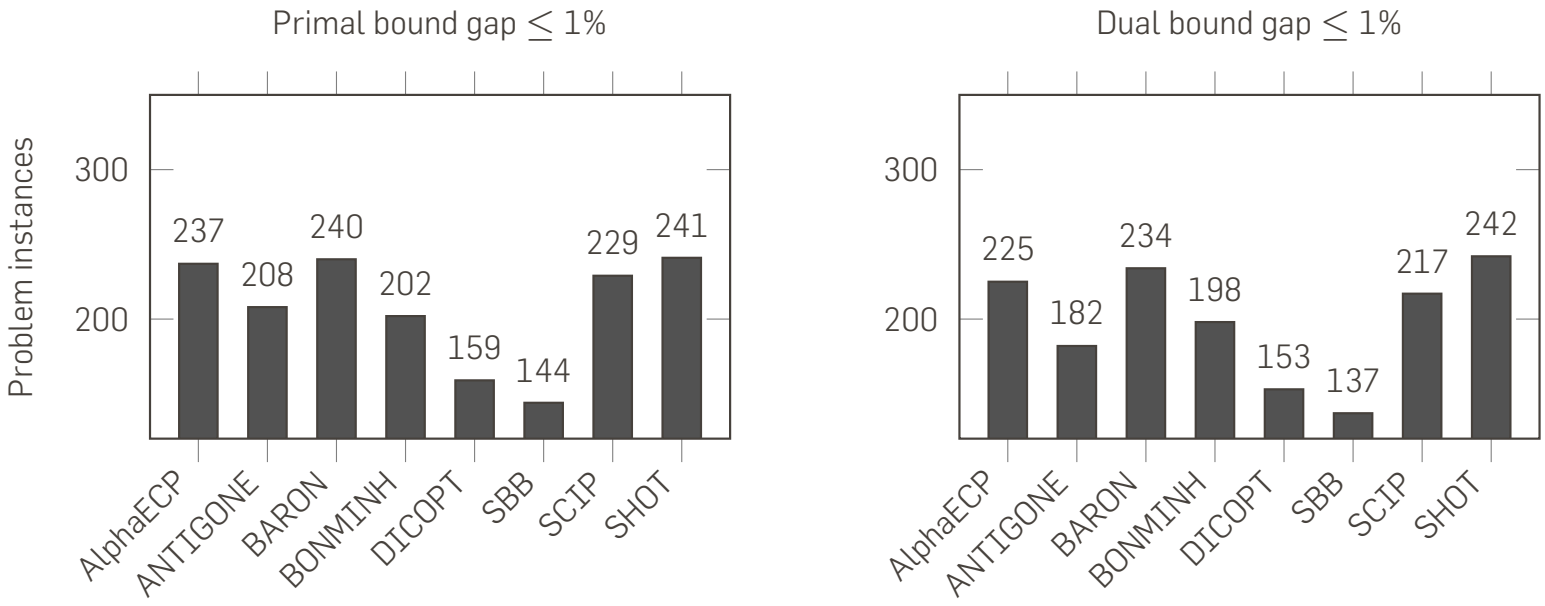
The total computational time for all solvers was about 150 h.

REASONS FOR TERMINATION



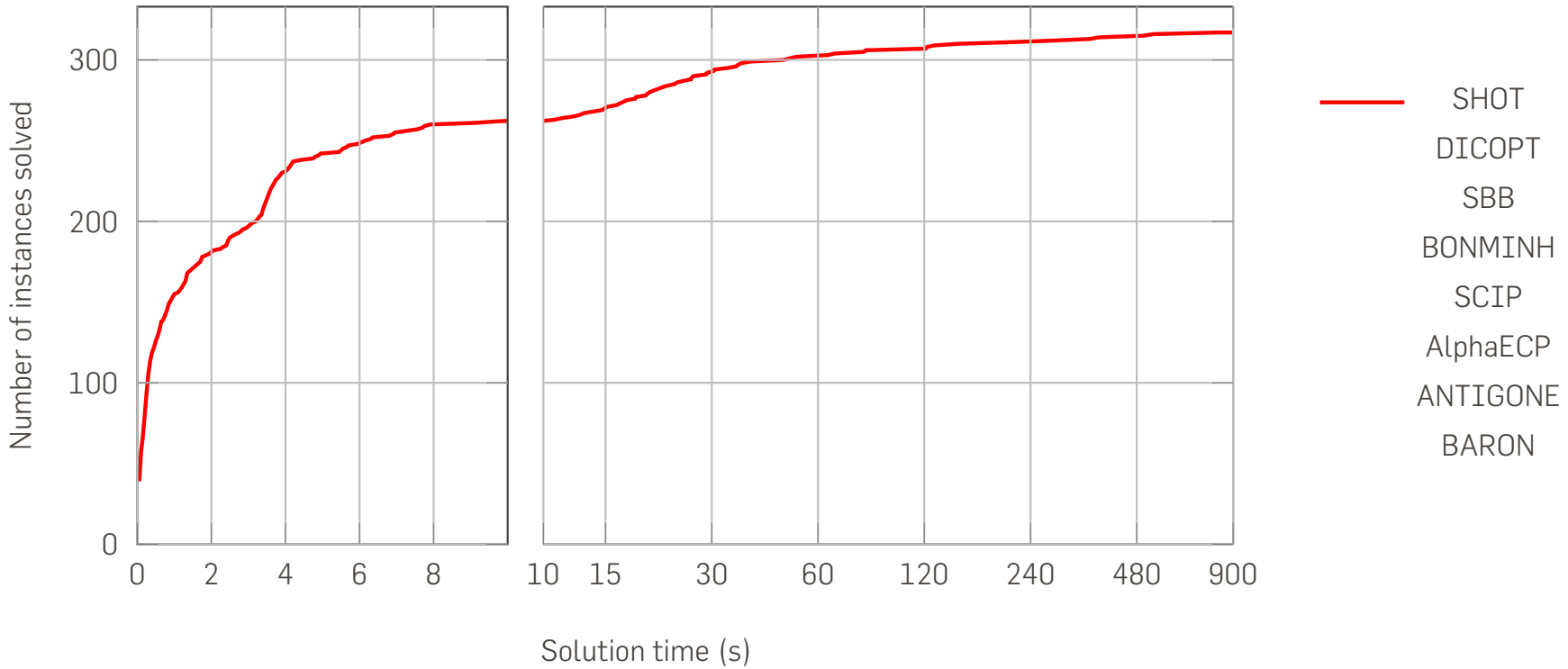
The termination statuses for the solvers as provided by PAVER 2. Note that a the solution can be optimal even though a limit (*e.g.*, time) has been reached, the solver has simply not proven optimality.

PRIMAL AND DUAL SOLUTION GAPS



Number of problem instances solved with primal and dual solution bound gaps $\leq 1\%$ as calculated by PAVER 2 for the 242 problems with reported optimal solution available in MINLPLib 2.

PERFORMANCE PROFILE



A performance profile of the number of problem instances solved to an objective duality gap $\leq 1\%$.

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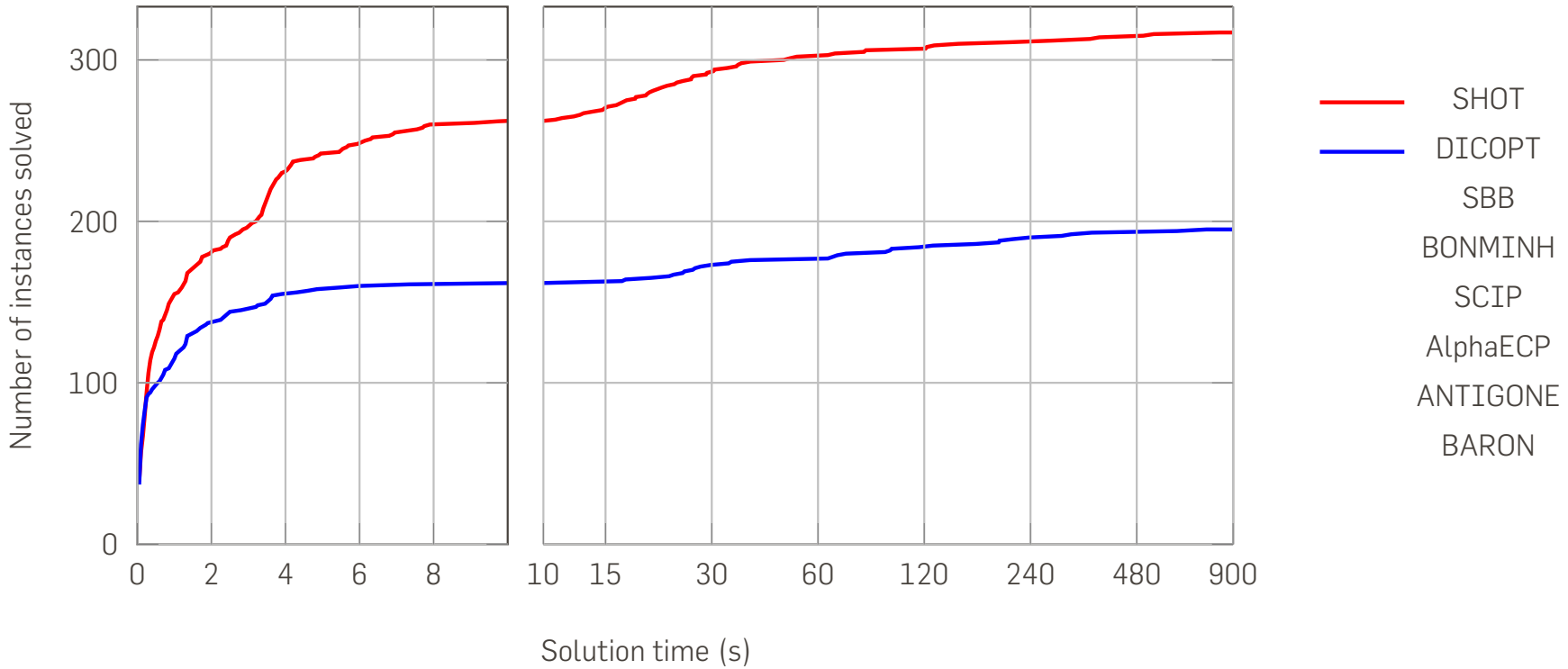

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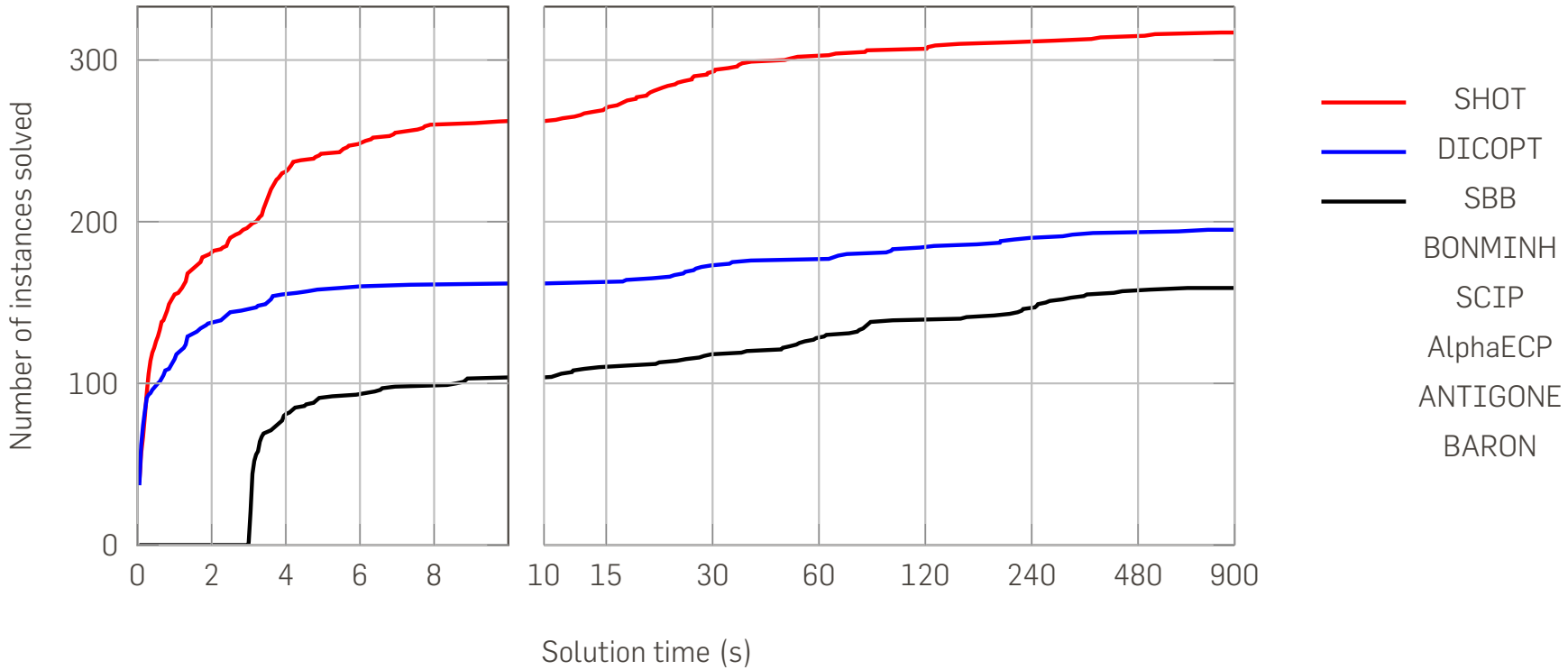

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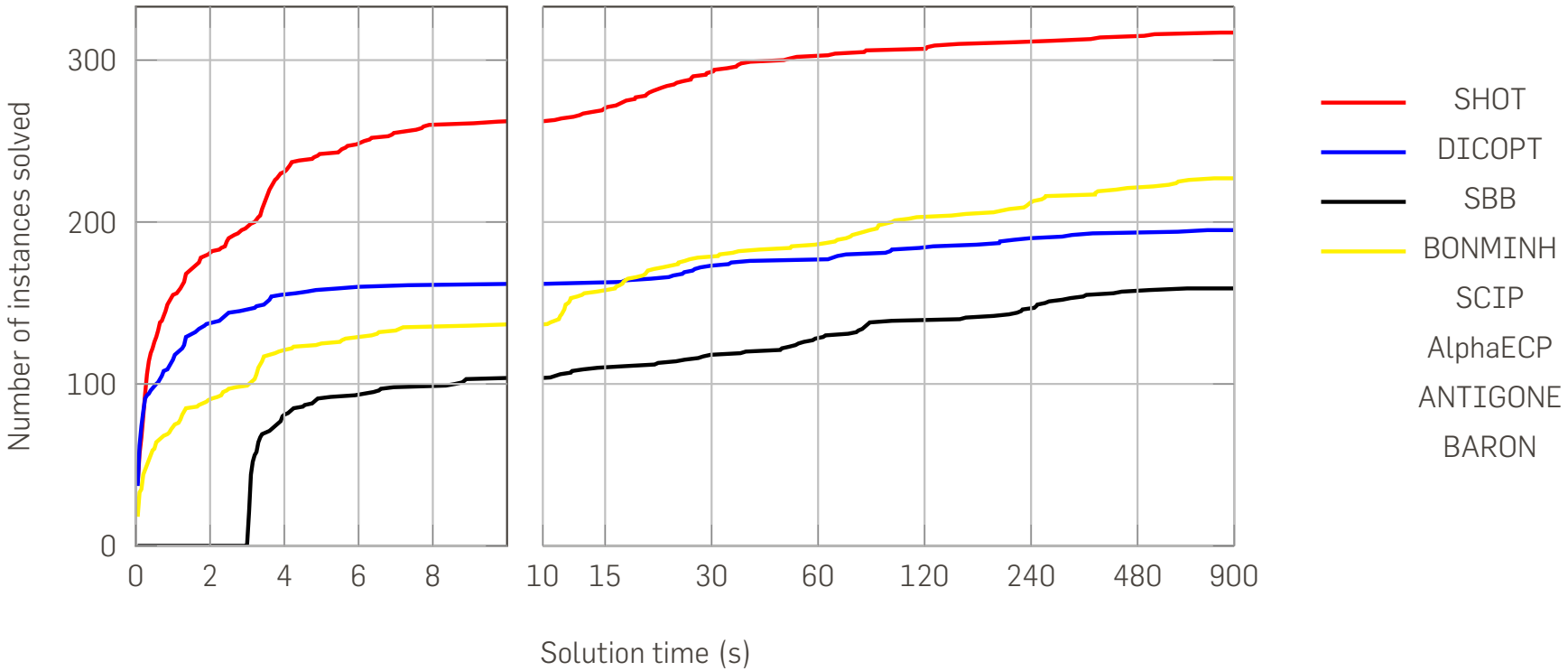

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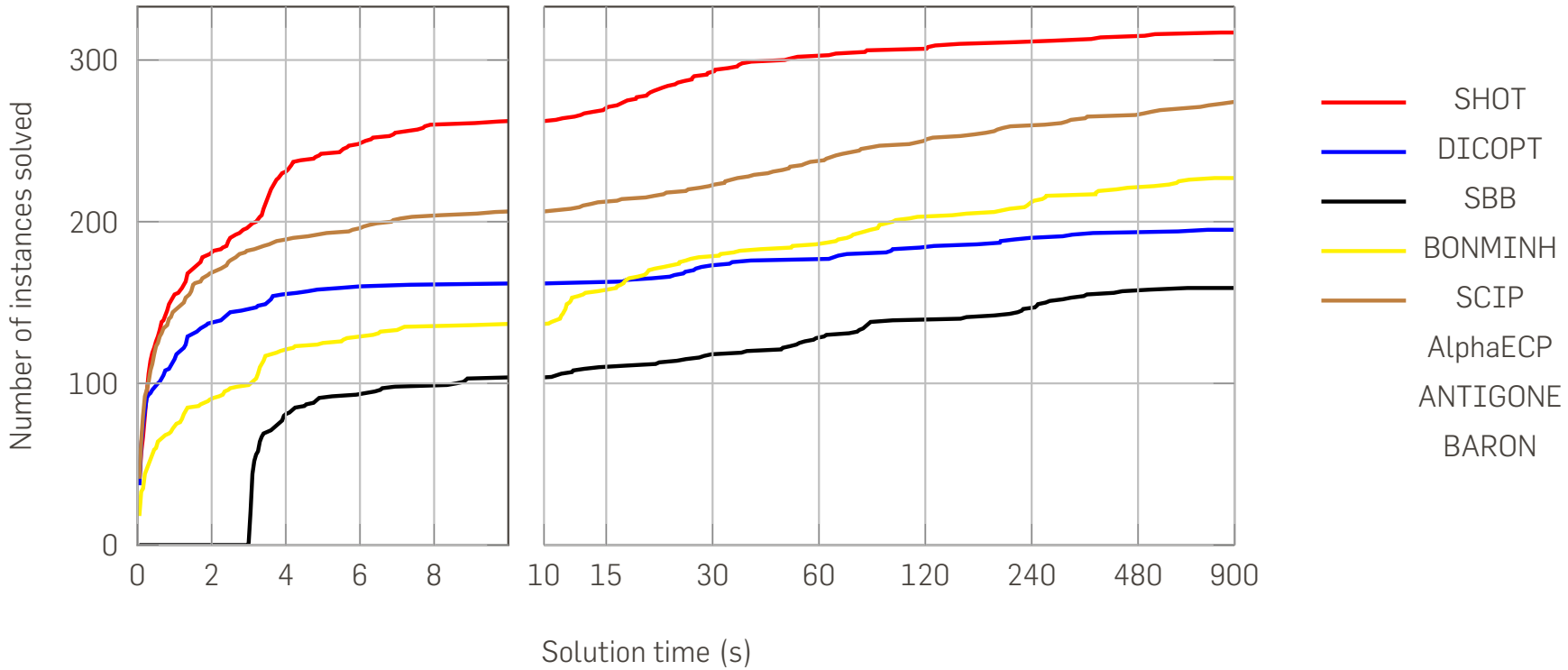

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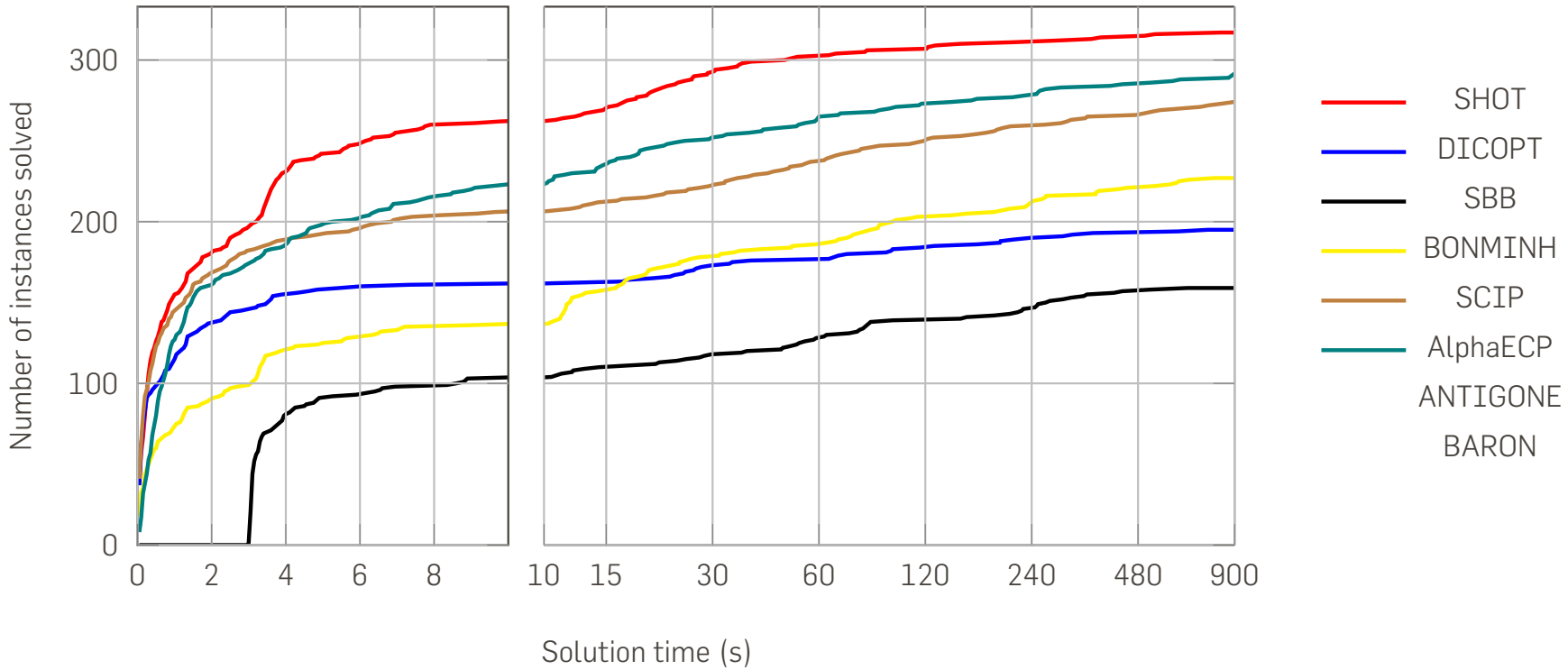

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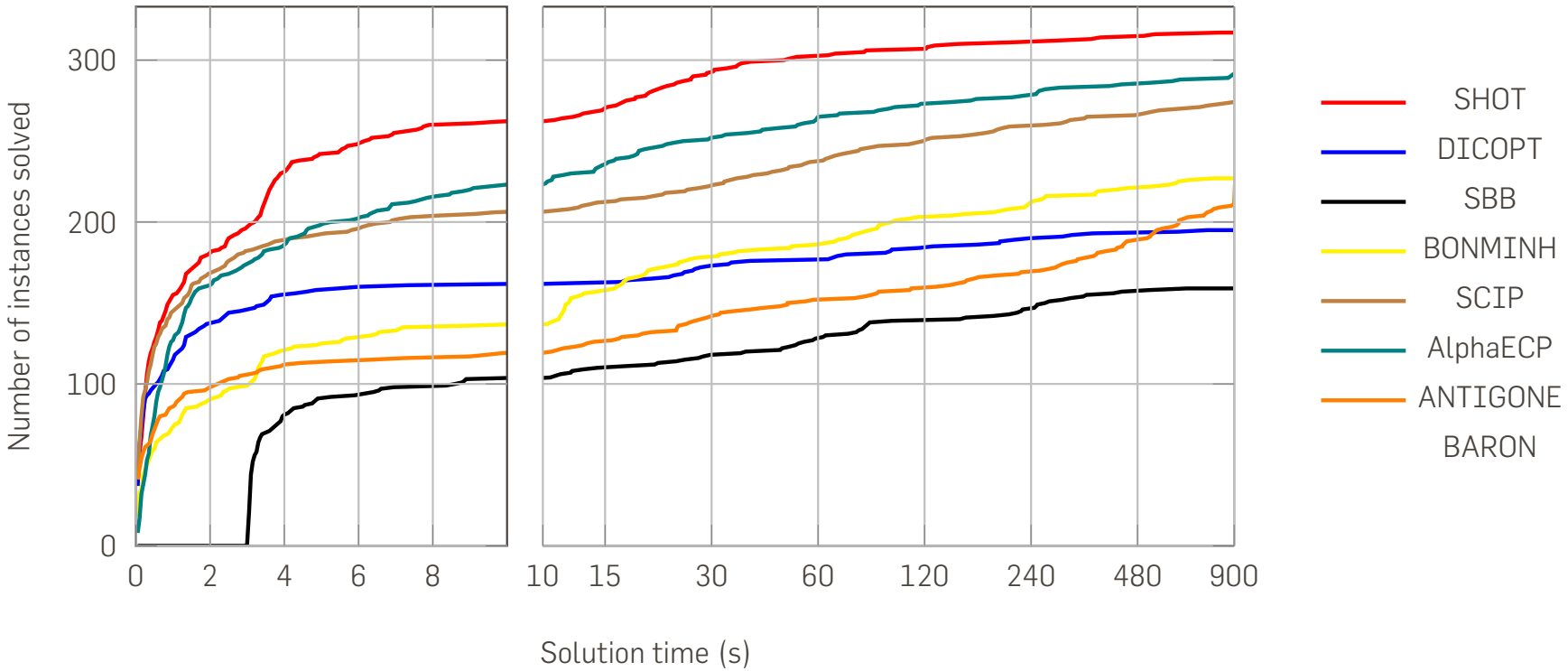

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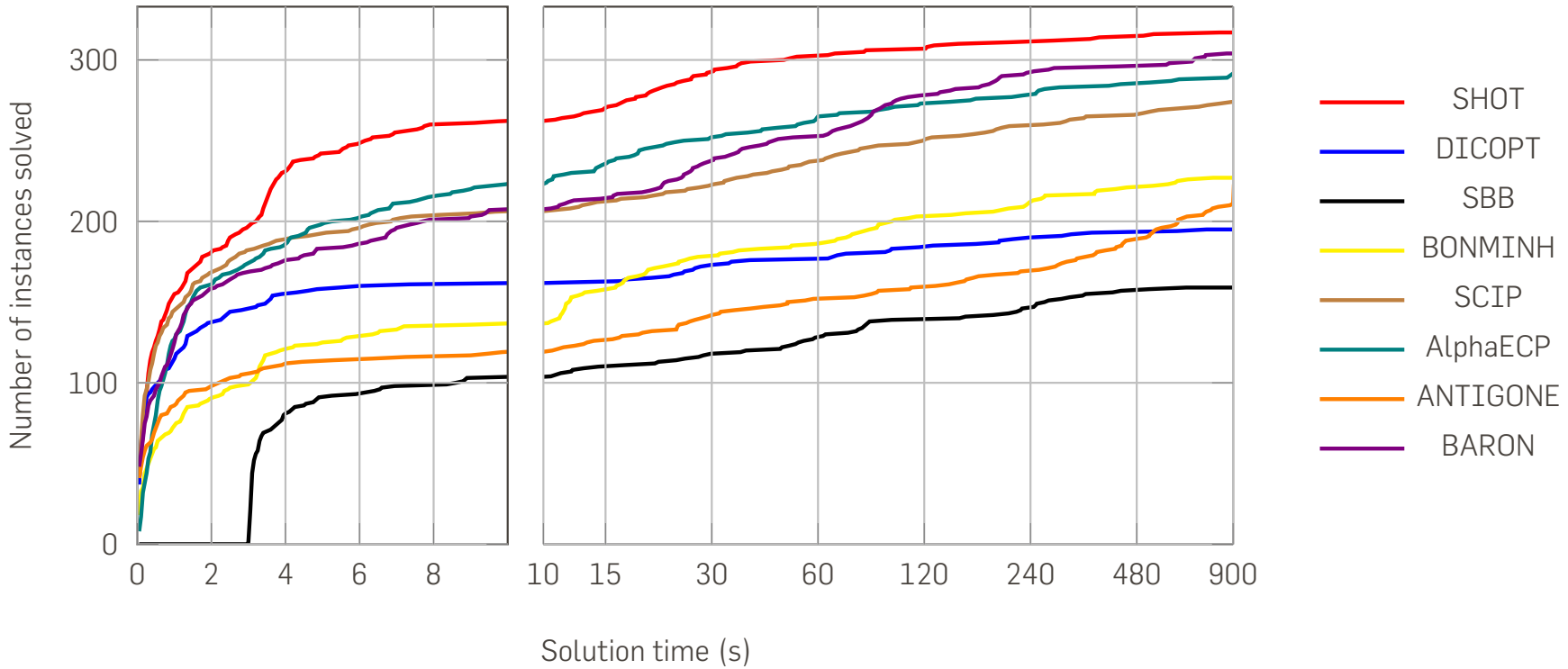

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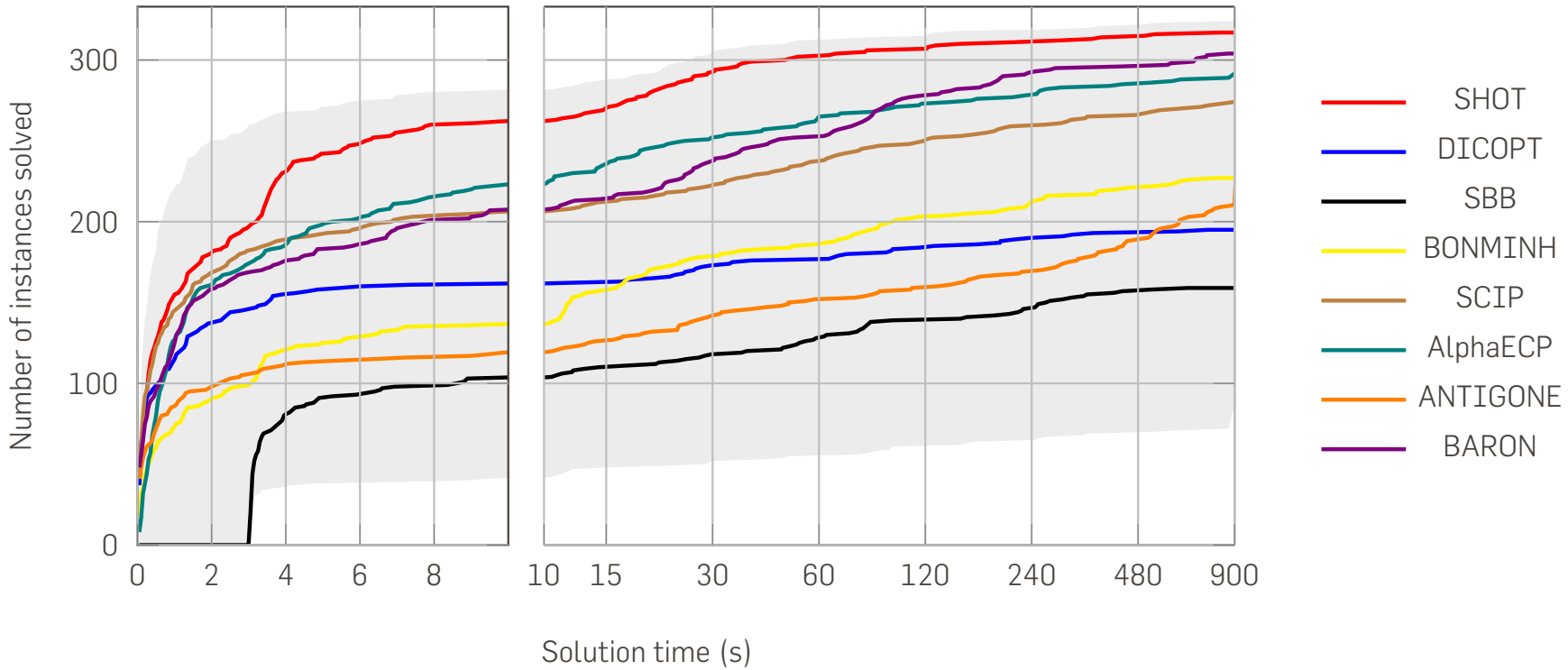

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Future research and development

- Investigate how to best select the interior point
- Improve handling of nonlinear objective functions
- Include the αSGO framework for global optimization of nonconvex MINLP problems

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Questions or comments?