Transformation-based and differential Equation-based Approaches for Non-convex MINLPs

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Outline

1. The Non-convex MIQP and the Linear Transformation
2. Numerical Comparison of the original and the Transformed Problem
3. Preprocessing before Convexification of Non-convex MIQP
   - Convex Reformulation of Non-convex MIQP
4. Different Equation based Approach for Solving General MINLP Problems

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$$\begin{aligned} \min_{x} \quad & h(x) = \frac{1}{2}x^T Hx + g^T x \\ \text{s.t.} \quad & Ax \leq b, \\ & Dx = e, \\ & l \leq x \leq u, \\ & x = \left(x_c^T, x_d^T\right)^T \in \mathbb{R}^{n_c} \times \mathbb{Z}^{n_d}, \\ & H \text{ is indefinite} \end{aligned} \tag{1}$$

The matrix H has the form

$$H = \begin{bmatrix} H_{cc} & H_{cd} \\ H_{cd}^T & H_{dd} \end{bmatrix},$$

$H_{cc} \in \mathcal{S}^{n_c}$, $H_{dd} \in \mathcal{S}^{n_d}$ and $H_{cd} \in \mathbb{R}^{(n_c,n_d)}$

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Let U_{dd} be the unimodular matrix

Let $x = Vy$ problem (1) is equivalent to

$$
\begin{aligned}
\min_{y} \quad & h(Vy) = \frac{1}{2} y^T V^T H V y + g^T V y \\
\text{s.t.} \quad & AVy \le b, \\
& DVy = e, \\
& l \le Vy \le u, \\
& y = \left[y_c^T, y_d^T \right]^T, \\
& U_{dd} y_d \in \mathbb{Z}^{n_d}, \\
& U_{cc} y_c + U_{cd} y_d \in \mathbb{R}^{n_c}.
\end{aligned} \tag{3}
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Problem (3) now takes the following form:

$$\begin{aligned} \min_{y} \quad & h(Vy) = \frac{1}{2}y^T V^T HVy + g^T Vy \\ \text{s.t.} \quad & AVy \leq b, \\ & DVy = e, \\ & l \leq Vy \leq u, \\ & y = \left[y_c^T, \, y_d^T\right]^T \in \mathbb{R}^{n_c} \times \mathbb{Z}^{n_d}. \end{aligned}$$

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& + U_{dd}^T H_{dd} U_{dd} \Big) y_d.
\end{aligned}
\tag{5}
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Motivation for the choice of U_{cc} and U_{dd}

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Lower bounding problem at each B&B tree under estimates each of bilinear term (one variable and two constraints)

$$\begin{aligned} y^T V^T H V y = & y_c^T U_{cc}^T H_{cc} U_{cc} y_c + 2 y_d^T \left(U_{cd}^T H_{cc} U_{cc} + U_{dd}^T H_{cd}^T U_{cc} \right) y_c \\ & + y_d^T \left(U_{cd}^T H_{cc} U_{cd} + U_{cd}^T H_{cd} U_{dd} + U_{dd}^T H_{cd}^T U_{cd} \right. \\ & \left. + U_{dd}^T H_{dd} U_{dd} \right) y_d . \end{aligned}$$

We set $\left(U_{cd}^T H_{cc} U_{cc} + U_{dd}^T H_{cd}^T U_{cc}\right) = 0$ **and see if** U_{cc} **and** U_{dd} **have desired properties**

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$$U_{cd} = -H_{cc}^{-1} H_{cd} U_{dd}.$$

Calculation of U_{dd}

$$\underset{(U_{dd})_i}{\operatorname{argmin}} \left\{ \max_{x_d} \left[\left(U_{dd}^{-1} \right)_i x_d : x \in \Omega_q \right] - \min_{x_d} \left[\left(U_{dd}^{-1} \right)_i x_d : x \in \Omega_q \right] \right\} \quad (7)$$

$$\text{s.t.} \quad \left(U_{dd}^{-1} \right)_{i,i} = \pm 1,$$
$$\left(U_{dd}^{-1} \right)_{i,j} = 0, \quad j = 1, \ldots, i-1,$$
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Calculation of U_{cc}

- H_{cc} is Hermitian it is diagonalisable. Let U_{cc} be the diagonalising matrix of H_{cc}.
- The columns of U_{cc} are the normalizing eigenvectors of H_{cc}.

$$
\begin{aligned}
y^T V^T H V y = & \, y_c^T U_{cc}^T H_{cc} U_{cc} y_c + 2 y_d^T \left(U_{cd}^T H_{cc} U_{cc} + U_{dd}^T H_{cd}^T U_{cc} \right) y_c \\
& + y_d^T \left(U_{cd}^T H_{cc} U_{cd} + U_{cd}^T H_{cd} U_{dd} + U_{dd}^T H_{cd}^T U_{cd} \right. \\
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\Theta_{dd} = U_{cd}^T H_{cc} U_{cd} + U_{cd}^T H_{cd} U_{dd} + U_{dd}^T H_{cd}^T U_{cd} + U_{dd}^T H_{dd} U_{dd} .
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$$
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The Transformed Problem

$$y^T V^T H V y = y_c^T \Theta_{cc} y_c + y_d^T \Theta_{dd} y_d \tag{9}$$

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$$\begin{aligned} \min_y \quad & h(Vy) = \frac{1}{2}\left(y_c^T \Theta_{cc} y_c + y_d^T \Theta_{dd} y_d\right) + g^T V y \\ \text{s.t.} \quad & AVy \le b, \\ & DVy = e, \\ & l \le Vy \le u, \\ & y^L \le y \le y^U, \end{aligned} \tag{10}$$

$$\begin{aligned} \min_x \quad & h(x) = \frac{1}{2} x^T H x + g^T x \\ \text{s.t.} \quad & Ax \le b, \\ & Dx = e, \\ & l \le x \le u, \end{aligned}$$

$$\min_{x} h(x) = \min_{y} h(Vy), \tag{11}$$

$$\operatorname*{argmin}_{x} h(x) = V \left\{ \operatorname*{argmin}_{y} h(Vy) \right\}. \tag{12}$$

Hessian Of the Transformed problem

$$\Theta = \begin{bmatrix} \Theta_{cc} & 0_{n_c \times n_d} \\ 0_{n_d \times n_c} & \Theta_{dd} \end{bmatrix}$$

Theorem

$h(x)$ has at most $\frac{1}{2}\left(n_c^2 - n_c\right) + n_c n_d$ more bilinear terms than $h(Vy)$

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2. Total problems used 384 when $n_c > n_d$
3. Total problems used 384 when $n_c < n_d$
4. If an algorithm ran for more than 10000 seconds on a problem it was stopped and declared unsuccessful

Figure: Performance profile obtained when solving problems with $n_c > n_d$ using SCIP

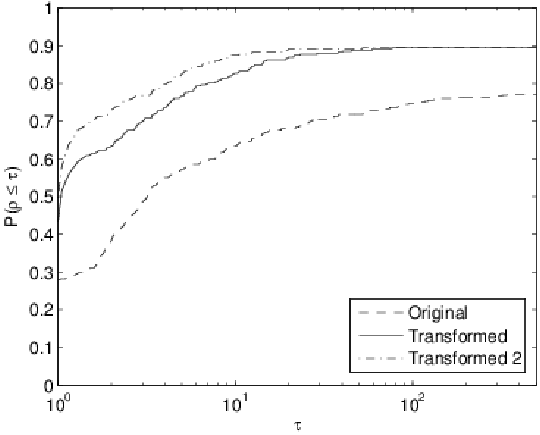


Figure: Performance profile obtained when solving problems with $n_c < n_d$ using SCIP

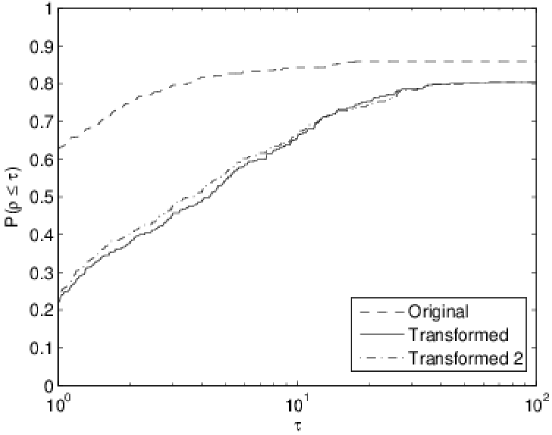
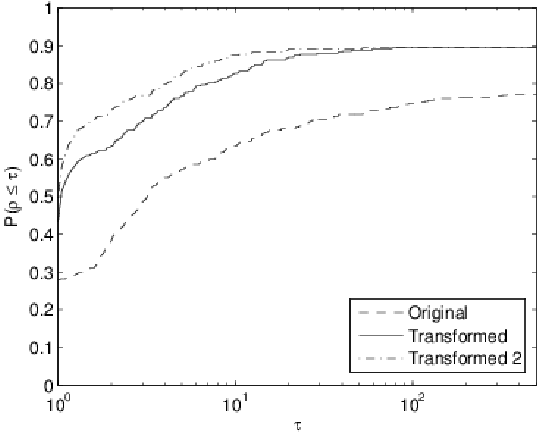
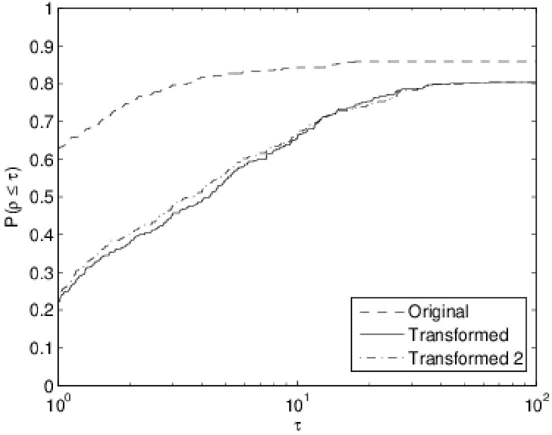


Figure: Performance profile obtained when solving problems with $n_c > n_d$ using BARON



Solution with $n_c < n_d$ using BARON

Figure: Performance profile obtained when solving problems with $n_c < n_d$ using BARON



Consider the convexification of the following non-convex MIQP

$$\begin{aligned} \min_x \quad & h(x) = \frac{1}{2}x^T Hx + g^T x \\ \text{s.t.} \quad & Ax \leq b, \\ & Dx = e, \\ & l \leq x \leq u, \\ & H_{cc} \quad \textcolor{red}{\text{Positive Semidefinite}} \end{aligned}$$

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Billionnet et al.(2012), Mathematical Programming 131, 381–401

Denote Convexification of above Problem as the Mixed Integer Quadratic Convex Reformulation (MIQCR)

Consider the Convexification of the following non-convex MIQP, H_{cc} Positive Definite

$$
\begin{aligned}
\min_{y} \quad & h(Vy) = \frac{1}{2}\left(y_c^T \Theta_{cc} y_c + y_d^T \Theta_{dd} y_d\right) + g^T V y \\
\text{s.t.} \quad & AVy \leq b, \\
& DVy = e, \\
& l \leq Vy \leq u, \\
& y^L \leq y \leq y^U,
\end{aligned}
$$

Denote Convexification of above Problem as the Mixed Integer Quadratic Transformation and Convex Reformulation (MIQTCR)

1. The reformulated MIQPs were solved using CPLEX 12.1 and the SDPs were solved using SeDuMi 1.3
2. If an algorithm ran for more than 10000 seconds on a problem it was stopped and declared unsuccessful

Figure: Performance profile comparing MIQCR and MIQTCR for $n_c = n_d$.

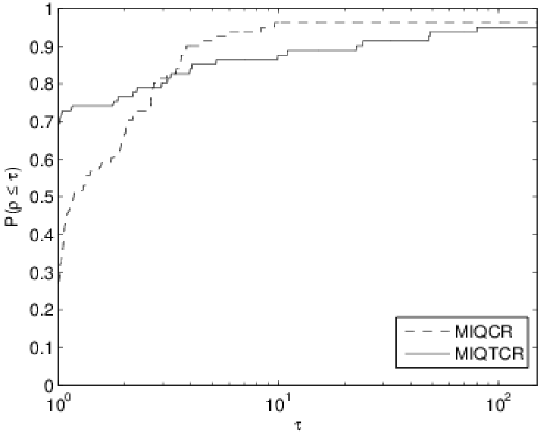
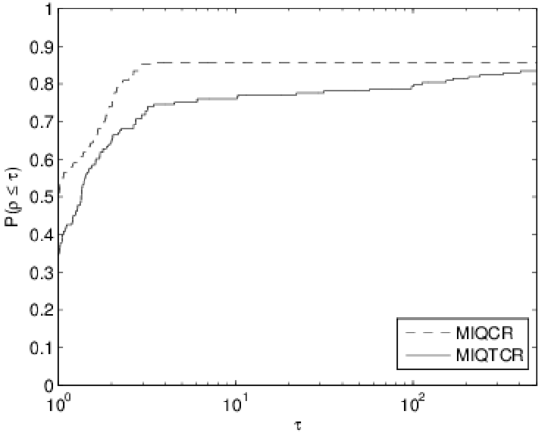


Figure: Performance profile comparing MIQCR and MIQTCR for $n_c < n_d$.



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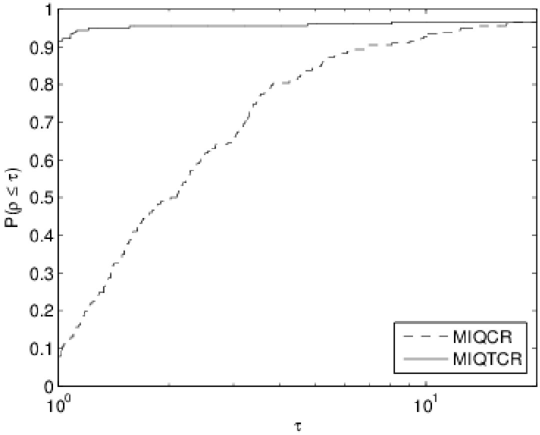

The non-convex terms of the transformed problem are bilinear terms involving only the integer variables.

$$V = \begin{bmatrix} U_{cc} & U_{cd} \\ 0_{n_d,n_c} & \widetilde{U}_{dd} U_{dd} \end{bmatrix}.$$

$$\begin{aligned} \min_{y} \quad & h(Vy) = \frac{1}{2}\left(y_c^T \Theta_{cc} y_c + y_d^T \Theta_{dd} y_d\right) + g^T Vy \qquad (13) \\ \text{s.t.} \quad & AVy \leq b, \\ & DVy = e, \\ & l \leq Vy \leq u, \\ & y^L \leq y \leq y^U \end{aligned}$$

Convex Reformulation by Pörn et al (1999):

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Applied to Our Transformed Problem (MIQTBC)

Convexification Results in a Convex MINLP – Not Convex MIQP

Results obtained using MINLP solver: Couenne 0.3.2 on the NEOS server

n	MIQCR	MIQTCR	MIQTBC
4	5.412	4.313	1.330
6	42.082	20.522	6.456
8	47.235	49.611	19.410
10	110.43	192.12	151.96
12	301.37	451.29	475.54
14	1032.1	1688.3	2012.5

Table: The time taken to solve problems using Couenne for Constraints Type 2

n	MIQCR	MIQTCR	MIQTBC
4	3.094	1.714	0.657
6	15.83	10.15	12.45
8	99.32	255.03	68.34
10	5352.3	3687.3	1958.6

Table: The time taken to solve problems using Couenne for Constraints Type 3

Transformation uses the following form of Hessian

$$\Theta = \Theta^{(1)} + \Theta^{(2)},$$

$$\Theta = \begin{bmatrix} \Theta_{cc}^{(1)} & 0 \\ 0 & \Theta_{dd}^{(1)} \end{bmatrix} + \begin{bmatrix} \Theta_{cc}^{(2)} & \Theta_{cd}^{(2)} \\ \Theta_{cd}^{(2)\,T} & \Theta_{dd}^{(2)} \end{bmatrix} \tag{14}$$

We have developed a B&B algorithm for solving this type of MIQPs

Figure: Performance profile when H_{cc} Singular using B&B for $n_c = n_d$.

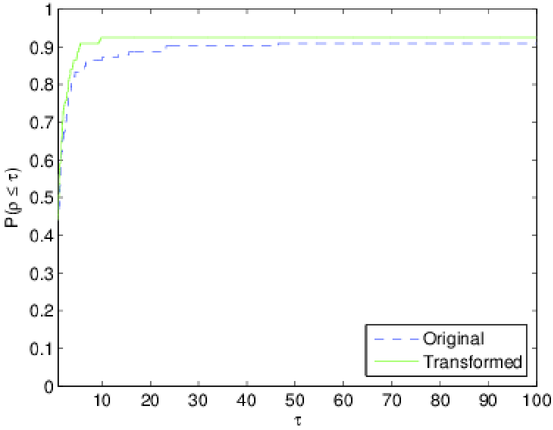
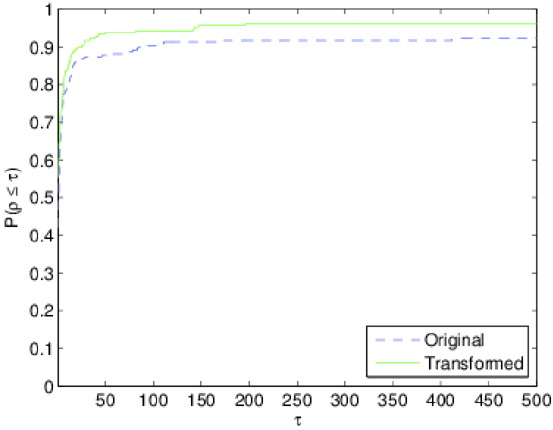


Figure: Performance profile when H_{cc} Singular using B&B for $n_c > n_d$.



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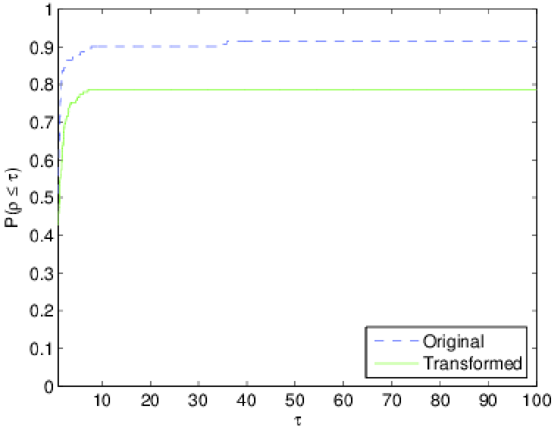

Background to the Trajectory-based Approach (Jan Snyman, 1982)

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Manipulate velocity such that

$$f(x^{(k+1)}) \leq f(x^{(k)})$$

Consider the general MINLP

$$(P)\begin{cases} \min\limits_{x,y} & f(x,y) \\ s.t. & g_i(x,y) \leq 0,\ i \in I \\ & g_j(x,y) = 0,\ j \in E \\ & x \in X, y \in Y\ integer, \end{cases}$$

Trajectory based Method uses (Ali & Oliphant 2014):

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iii) Augmented Lagrangian using fixed Integer Variable

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Feasible Continuous Manifold: Constrained Region over which $f_M(x)$ is defined.

Manifold Minimum*: Minimum of $f_M(x)$ on the *Feasible Continuous Manifold

i) Only Equality Constraints:

$$L(x, \lambda; \mu) = f(x) - \sum_{i \in E} \lambda_i g_i(x) + \frac{\mu}{2} \sum_{i \in E} c_i^2(x),$$

The Augmented Lagrangian Function for (P)

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ii) Only Inequality Constraints:

$$L(x, s_i, \lambda; \mu) = f(x) - \sum_{i \in I} \lambda_i (g_i(x) + s_i) + \frac{\mu}{2} \sum_{i \in I} (g_i(x) + s_i)^2, \ s_i \geq 0$$

iii) Only Equality & Inequality Constraints:

$$\phi_A(x,\lambda;\mu) = f(x) - \sum_{i\in E} \lambda_i g_i(x) + \frac{\mu}{2}\sum_{i\in E} g_i^2(x) + \psi(x,\lambda;\mu)$$

where

$$\psi(x,\lambda;\mu) = \begin{cases} \sum_{i\in I} -\frac{\lambda_i^2\mu}{2}, & g_i(x) - \mu\lambda_i \geq 0, \\ \sum_{i\in I} -\lambda_i g_i(x) + \frac{1\mu}{2} g_i^2(x), & g_i(x) - \mu\lambda_i \leq 0. \end{cases}$$

$$\ddot{x} = -\nabla_x \phi_A(x, y, \lambda; \mu), \quad x(0) = x_0, \quad \dot{x}(0) = 0$$

Trajectory-based Approach for General MINLP Problems

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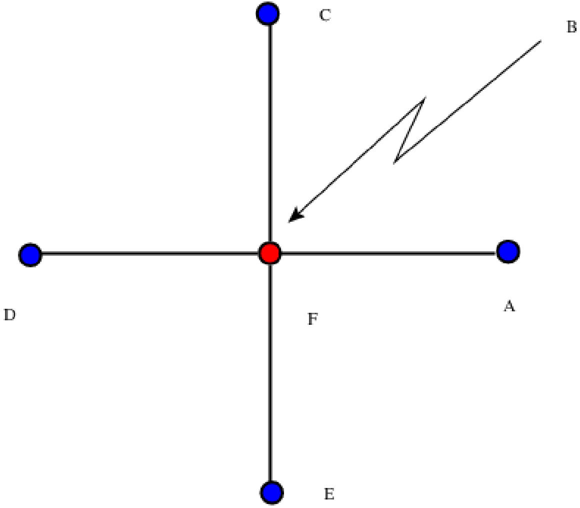
Discrete Neighbourhood

$$\mathcal{N}_r(x) = \{y \in \mathbb{R}^n : y_c = x_c, \; \|y_d - x_d\| \leq 1\}.$$

Figure: Starting Solutions for Manifold Minimization



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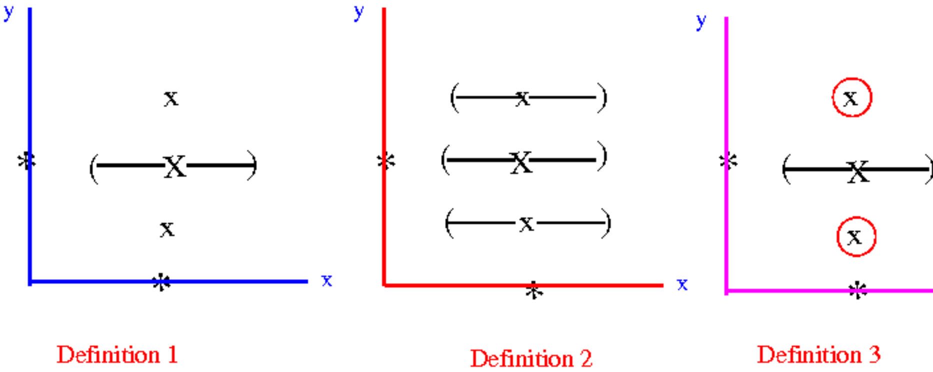

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P	(n_c)	(n_d)	f(x)	LE	LI	NE	NI	k_1	k_2	fe	$\lvert\epsilon_d\rvert$
1	1	1	Linear	0	4	0	1	I	0	0	1
2	1	1	Linear	0	5	0	1	I	147	0	1
3	1	1	Linear	0	4	0	0	56	43	3	1
4	1	1	Linear	0	4	0	2	94	98	3	1
5	1	1	Bilinear	0	6	0	0	105	0	1	1
6	1	1	Nonlinear	0	4	0	0	113	40	0	1
7	1	1	Nonlinear	0	3	0	1	34	18	3	1
8	1	1	Nonlinear	0	5	0	1	150	41	3	1
9	1	1	Nonlinear	0	5	0	1	21	0	3	1
10	1	1	Nonlinear	0	6	0	0	I	218	3	1
11	1	2	Linear	0	6	0	1	130	E	-	-
12	2	1	Linear	0	1	0	5	108	0	1	1
13	1	2	Linear	0	4	0	0	I	102	5	1
14	2	1	Nonlinear	0	8	0	1	129	212	1	1
15	2	2	Bilinear	10	0	0	0	I	264	7	1
16	1	3	Nonlinear	12	0	0	0	46	E	-	-
17	1	3	Nonlinear	0	12	0	0	I	488	6	2
18	2	3	Linear	1	12	0	1	I	452	4	1
19	3	3	Linear	12	0	2	0	222	0	0	1
20	3	3	Nonlinear	0	10	0	2	I	343	4	1
21	3	3	Nonlinear	0	16	0	2	401	179	4	1
22	3	4	Nonlinear	0	12	0	5	I	291	4	1
23	2	6	Linear	0	21	0	1	D	D	-	-
24	7	2	Linear	3	14	2	0	I	285	3	1
25	3	8	Linear	0	26	3	0	I	190	8	1
26	6	5	Nonlinear	21	0	3	0	D	D	-	-
27	6	5	Nonlinear	1	32	0	3	I	382	6	1
28	9	8	Nonlinear	2	51	0	4	I	387	7	1

Thank You!